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CONTENTS

Geology. — *On the age of the serpentines in Cuba.* By L. RUTTEN.

(Communicated at the meeting of April 27, 1940.)

In 1933 the author has explored with four students of the Utrecht University different parts of Cuba. The results of these studies have been published in four academical theses (4, 8, 9, 12), in which the geology of Pinar del Rio, of North and South Santa Clara and of a large part of Camaguey has been described. In these four areas serpentines have been found. The data obtained in Pinar del Rio were not sufficient for fixing the age of the serpentines in this province. The serpentines of Santa Clara and Camaguey are pre-upper-cretaceous. The same can be stated with great probability for the serpentine of Guanabacoa, E. of Habana, which the author studied almost twenty years ago (7). The arguments are the following:

Northern Santa Clara. There occur dikes and other intrusions of quartz-dioritic rocks in the serpentine, which therefore must be older. Pebbles of quartzdioritic rocks are found in the upper-cretaceous Habana-formation: the Habana-formation, therefore, is younger than the quartzdioritic rocks. The serpentines, being older than the quartzdiorites, are pre-upper-cretaceous.

Southern Santa Clara. Serpentine has been found: 1. as serpentine-schist, intercalated in the schists of the Trinidad Mountains; these schists are jurassic or older; 2. in a very small exposure, comparable with the serpentines of Northern Santa Clara.

Camaguey. The serpentines are older than quartzdioritic rocks, which are intrusive in the serpentines. Pebbles of quartzdioritic rocks occur in the upper-cretaceous Habana-formation. Pebbles of serpentine have been found in the Habana formation and in the eocene Cubitas-limestone. The serpentines are pre-upper-cretaceous.

Guanabacoa. A dike of quartzdiorite in serpentine has been found. It is therefore supposed — by analogy — that the serpentines have the same age as in Santa Clara and Camaguey.

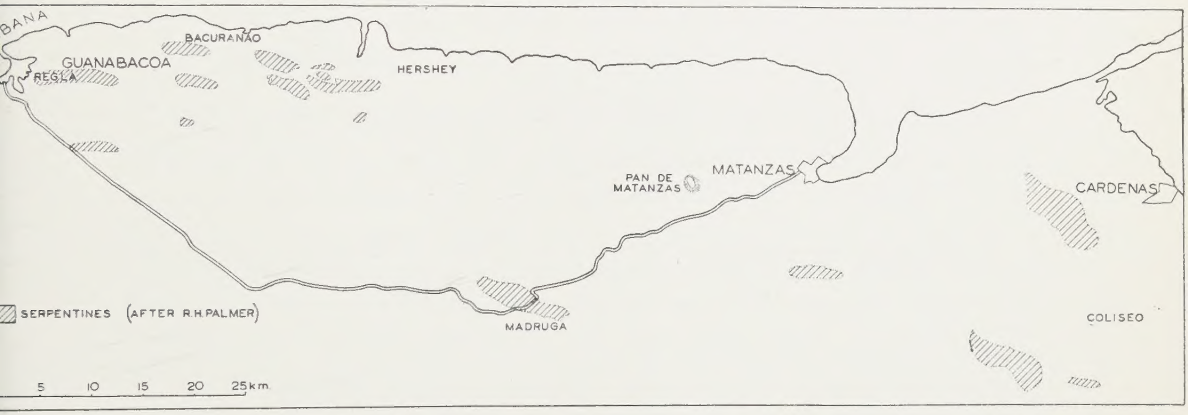
In 1939 the author studied with six students of the Utrecht University large areas in E. Santa Clara and W. Camaguey, and in E. Camaguey and Oriente. Again we met with many areas of serpentine. The study of the rock-collections has hardly begun, but it is possible to state, on the base of our field-observations, that all the serpentines are pre-eocene and very probably pre-upper-cretaceous.

There are other geologists who feel certain that at least part of the Cuban serpentines is much younger, and more specially that some serpentines in Matanzas are post-miocene (post-"Güines-limestone"). The spokesman of these geologists is R. H. PALMER (6). Dr. PALMER, whom I had the pleasure to know already in 1933, was so kind as to accompany me in 1939 to two localities in Matanzas where, according to him, strong proof was to be found for the youth of the serpentines.

On the base of the field observations and still more on the base of the study of the collected rock-material I have reached the conviction *that these serpentines are certainly pre-miocene.*

The first locality lies about 10 km to the South of Matanzas, on the automobile-road from Matanzas to the S. PALMER gives the following description:

"At kilometer 13.5 on the highway a serpentine intrusion crosses the road. This is a locality of considerable importance as the contact relationships between the serpentine and the Güines limestone are in evidence. It may be noted that the Güines limestone is fractured in the vicinity of the contact, that limestone fragments are imbedded in the serpentine and that near the serpentine the limestone is crystallized in contrast to its amorphous condition 100 yards distant from the intrusion. This locality is taken as evidence that there are post-lower Miocene serpentine intrusions" (6).



We could not find fragments of limestones, imbedded in the serpentine. We collected samples of the serpentine and of the limestones quite near to the contact; moreover we took samples to the North of the contact at different distances, and as far as about 100 m. The description of the samples follows here. The contact is to be seen in a small quarry. W. of the highway.

The serpentine passes near the contact into a rock which contains totally weathered and rounded serpentine-fragments in a "groundmass" of dolomite. I suppose that this rock is the basal conglomerate of the Güines limestone; it is, however, very strongly weathered, and I leave it therefore out of consideration.

A rock, resembling the former, but containing less serpentine, has been sampled at a distance of 0.25 m above the former.

A white, crystalline dolomite, poorly porous, and containing fine rhomboëdrons, proceeds from a zone, 0.75 m higher than the former.

A porous, brown dolomite with a limonitized crystal of magnetite and with *rounded grains of serpentine* has been sampled 0.50 m higher in the section.

The last sample of the section, from a zone, about 1 m above the former, is a strongly porous dolomite, which contains many beautiful rhomboëdrons.

As an isolated fragment there has been collected in the quarry another brown, porous dolomite, *containing a small inclusion of serpentine.*

A dozen samples have been taken along the roadside at distances of about 2, 12, 25,

30, 42, 50, 55, 65, 80, 84, 96 and 100 m to the N. from the contact. The first 7 samples are porous dolomites, which always present in the slide beautiful sections of numerous rhomboëdrons; they are very similar to the last mentioned samples from the quarry. Farther to the N. the rocks become less dolomitized and pass into normal limestones.

Thus, it is indeed true that the crystallinity of the carbonatic rocks diminishes from the contact with the serpentine to the N., but this has apparently nothing to do with contactmetamorphism. It is a phenomenon, we meet with in all limestone-areas with intercalated dolomitized zones, even if intrusive rocks are absolutely absent (f.i. Dalmatia, Ardennes, Jura). *There is no trace of contact minerals in the dolomite close to the contact, and the occurrence of rounded grains of serpentine in the dolomites proves that the Güines-limestones are younger than the serpentine.*

The occurrence of the very thin layer of poorly porous, white dolomite close to the contact is rather remarkable; it was formed probably by descendant carbonatic solutions, from which the dolomite crystallized near the contact with the impermeable serpentine.

The second contact of serpentine and Güines limestone which we visited, is on the automobile-road from Matanzas to Habana, about 25 km to the W. of Matanzas. PALMER (6) writes with regard to this outcrop:

"Observers are not in accord as to the nature of this contact. On one side it is said that the limestone shows incipient crystallization next to the serpentine and that small pyrite crystals occur in this narrow zone, both of which are taken as evidence of contact metamorphism and further there is no trace of serpentine pebbles in the contiguous limestone as would be expected if the limestone were deposited on a serpentine base. The opponents of this theory deny there is any trace of contact metamorphism......"

At this locality the serpentine is separated from the limestone by a more or less sandy layer of 1—2 m thickness. We have taken a sample of the sandy layer and two of the limestones.

The limestone nearest to the serpentine is greyblue; its colour becomes brown in consequence of weathering. In the slide it presents itself as a very finely crystalline rock with small veins of calcite.

The limestone farther from the contact is more strongly crystalline; thus there is apparently no relation between crystallinity and distance from the contact.

Various slides of the sandy rock have been made. It appears to be a calcareous sandstone which contains numerous well-rounded grains, belonging to the following rocks: 1. strongly weathered groundmass of porphyrites, 2. quartz, 3. mikropegmatite, 4. ?talc and 5. strongly weathered *serpentine* [1]).

Thus, the sedimentation of the Miocene begins here with a basal sandstone which contains grains of serpentine; the Miocene is therefore certainly younger than the serpentine.

Dr. PALMER mentions another serpentine-mass, to the S. of Cardenas, which he also considers apparently as intrusive into the Güines-limestone.

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[1]) The slides of the above-described rocks in the Utrecht collection have the numbers 19200—19225.

He does not give, however, any arguments for this locality. As I did not visit it, I cannot discuss it here.

PALMER seems to be of the opinion that all the other serpentines in the provinces of Habana and Matanzas are post-upper-cretaceous {post-Habana-formation (6)}. He mentions repeatedly the existence of serpentine-intrusions in the belt of Cretaceous between Habana and Matanzas. He says of a serpentine-mass under the Pan the Matanzas:

"one large intrusion running in a northwest-southeast direction passes under Pan de Matanzas and from the air appears to have elevated this prominent hill" (6).

Describing the aspect of Yumurí-valley from an elevated point he writes (6):

"The floor of the valley is upper Cretaceous interrupted by long, narrow tongues of serpentine intrusions".

And from Barreras, to the E. of Habana he says:

"Beyond Barreras a few tongues of serpentine intruding the Cretaceous cross the road".

Here, again, PALMER's opinion does not harmonize with our results in Santa Clara, Camaguey and Oriente. I am sorry not to have been able to visit the different serpentines between Habana and Matanzas, but I have restudied the slides of the cretaceous [1]) rocks which I collected long ago to the East of Habana Bay.

At the S. side of the serpentine of Guanabacoa (E. of Habana Bay) there is a visible contact between serpentine and cretaceous limestone. The latter dips strongly to the S. It contains (slide 8444) Camerina dickersoni and Gumbelina; *it does not show the slightest trace of contact-metamorphism.*

A sandy limestone at the S. side of the serpentine of Guanabacoa (slide 8450) contains *rounded grains of serpentine.*

A calcareous sandstone, collected at a small distance to the N. of the serpentine of Guanabacoa (slides 8861, 8863) contains tuffaceous material and moreover *grains of serpentine.*

A limestone from Regla, at the E. side of the Bay of Habana, (slide 8849) contains Camerina dickersoni, Gumbelina and *rounded grains of serpentine.*

A sandy limestone with Vaughanina cubensis from Luyanó, S. side of the Bay of Habana, contains *rounded grains of serpentine* (slide 8843).

A sandy limestone from Bacuranao (slide 8479), about 15 km to the E. of Habana Bay, contains *rounded grains of serpentine.*

All these observations prove that the serpentines which occur E. of Habana Bay, are pre-upper-cretaceous.

In consequence there is no evidence for a different age of the serpentines of the provinces of Habana and Matanzas and those of Santa Clara, Camaguey and Oriente. It would, indeed, be very remarkable, if such rare

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[1]) In my publication of 1922 (7) I made a blunder in considering the cretaceous rocks of the Habana formation as Older Tertiary. I found Camerina's and what I considered to be Lepidocyclina's. The first are "cretaceous Camerina's" (C. dickersoni), the second are Lepidorbitoides, Orbitoides and Vaughanina. See also (5).

and "aberrant" rocks as serpentines had been formed in so small an area as Cuba in two different geological periods.

It has been advocated that a relation exists between the distribution and age of the West Indian serpentines and problems of fundamental geological importance. H. H. HESS (2) has established the existence of a belt of strong gravity anomalies in the West Indies, quite comparable with the well-known belt of negative gravity anomalies in the East Indies. The existence of such a belt in the Antilles had already been made probable by VENING MEINESZ (10, 11). It needs not to be said that this discovery is of very great importance; the present author feels certain that in some way these zones of negative gravity-anomalies will play a great part — as they have already done — in speculations on orogenetic processes. He cannot, however, agree with H. HESS, who has tried to find a plausible relation between the zone of gravity-anomalies, the origin of the serpentines and orogenetic processes. The ideas of HESS may be summed up about as follows (1, 2). The zone of negative gravity anomalies is explained by him — in harmony with VENING MEINESZ (10, 11) — by a downward buckling of the earth's crust to a depth of 40—60 km. Strong folding of the crust's superficial layers will come into existence above the down-buckle, and peridotitic-serpentinic magma will be squeezed-up and form intrusions in the neighbourhood of the axis of the downbuckle. For the West Indies this would have the following consequences: 1. the serpentines are to be found on the axis of the down-buckle or very near to it, 2. the strongest deformations are to be found on the axis of the down-buckle, 3. the serpentines must be younger than the strongly folded formations. Now, the following may be observed.

Serpentines are well-known from Cuba and Porto Rico. In Cuba they are indeed more or less clearly localized on the axis of the zone of negative gravity-anomalies, which, however, is not well-pronounced on the island. In Puerto Rico (3), the serpentine-mass of Mayaguez which is the only important one, is situated at the enormous distance of about 150 km from the axis of negative-gravity-anomalies.

It is not possible to check for the West Indies, whether the zone of strongest crustal movements coïncides with the axis of the gravity anomalies as, for the greatest part, this axis passes over the sea.

According to HESS the negative strip originated in the Eocene. As we have seen that the serpentines of Cuba are pre-upper-cretaceous it is clear that the mechanism, constructed by HESS, cannot have any reality, at least, not in its present form.

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Mathematics. — *Zur projektiven Differentialgeometrie der Regelflächen im R_4.* (Zweite Mitteilung). Von R. WEITZENBÖCK.

(Communicated at the meeting of April 27, 1940.)

Ich untersuche hier die mit einer allgemeinen Erzeugenden der Fläche F projektiv-invariant verknüpften linearen Räume und deren gegenseitige Beziehungen. Zwecks leichterer Verweisung sind Paragraphe und Formeln im Anschlusse an die erste Mitteilung durchnummeriert.

§ 4.

Wir setzen im Folgenden voraus, dass F nicht abwickelbar, dass also

$$(x M'_{02}) = (x 0^2\, 2^2) = (x a^2\, a_2^2) \not\equiv 0 \quad \dots\dots \quad (31)$$

sei. $(x M'_{02}) = 0$ stellt den *Tangentialraum an F längs der Erzeugenden* 0_{ik} dar als Verbindungsraum zweier benachbarter Erzeugender.

Die Schnittebene m_{ijk} zweier aufeinanderfolgender Tangentialräume hat wegen (1) die Gleichung

$$\pi_{02,03} = (\pi\, 0^2\, 2^2)(\pi\, 0^2\, 3^3) = 0 \;,\; m'_{rs} = (M'_{02}\, M'_{03})_{rs} \,. \quad . \quad . \quad (32)$$

Wir werden sie im Folgenden „*Heftebene*" m_{ijk} nennen. Da

$$\pi_{02,03} = -2_{0\pi,03} = -(2 0^2 \pi^2)\, 2_{03}$$

können wir auch setzen

$$m_{ijk} = (0^2\, 2)_{ijk}\, 2_{03} \quad \dots\dots\dots \quad (33)$$

Die Heftebene wird unbestimmt, wenn der Tangentialraum M'_{02} stationär ist. Gilt dies für jede Erzeugende, so liegt F ganz in einem R_3. Es ist dann

$$(\pi^2\, 0^2\, 2)\, 2_{03} \equiv 0 \quad \{\text{für alle } \pi_{ik}\} \quad \dots\dots \quad (34)$$

und dies lässt sich wie folgt vereinfachen:

$$(\pi^2\, 0^2\, 2)\, 2_{03} = -3 \,.\, (\pi^2\, 0^2\, 2)\, 2_{12} = + \tfrac{3}{2}\, (\pi^2\, 0^2\, 1)\, 1_{22} = - \tfrac{3}{2}\, (\pi^2\, 1^2\, 0)\, 0_{22}$$

Führen wir also nach (21) den „*Heftpunkt*" H

$$H = (H u') = (0\, u')\, 0_{22}$$

ein, so kann die Heftebene auch so dargestellt werden

$$m_{ijk} = -\tfrac{3}{2}\, (1^2 H)_{ijk} \,, \quad \dots\dots\dots \quad (35)$$

d.h. die Heftebene ist die zum Punkte H gehörige Nullebene bezüglich des Ebenenkomplexes 1_{ik}. (Jede Gerade, die die Nullebene schneidet, gibt mit H verbunden eine Ebene des wegen $M'_{02} \neq 0$ nicht-speziellen Ebenenkomplexes 1_{ik}). (34) ist also ersetzbar durch $(Hu') = 0$ in Uebereinstimmung mit dem am Ende des vorigen Paragraphen gegebenen Kriteriums.

Drei aufeinanderfolgende Tangentialräume schneiden sich in einer Geraden mit der Gleichung

$$(\pi M'_{02})(\pi M'_{03})[(\pi M'_{13}) + (\pi M'_{04})] = 0 \quad \ldots \quad (36)$$

Mit Hilfe von (1) lässt sich dies schreiben als

$$\tfrac{3}{4}(\pi^2 0^2 2)\, 2_{03}(\pi_{04} - \pi_{22}) = 0$$

und dies kann umgeformt werden zur Gleichung der Geraden

$$m_{ik} = a_{ik} + 4Q \,.\, 0_{ik} = (0\,2)_{ik}\, 0_{22}\, 2_{03} + 4Q \,.\, 0_{ik} \quad \ldots \quad (37)$$

Die hier auftretende Differentialinvariante Q ist durch (17) gegeben.

Die in (37) stehenden Linienkoordinaten

$$a_{ik} = (0\,2)_{ik}\, 0_{22}\, 2_{03} = (H2)_{ik}\, 2_{03} \quad \ldots \quad (38)$$

geben die „*Heftgerade*“ [1]) der Erzeugenden $0_{ik} = a_{ik}$. Sie schneidet diese Erzeugende im *Heftpunkt* H und liegt mit ihr in der *Heftebene* m_{ijk}. Die Heftgerade a_{ik} verbindet den Heftpunkt H mit dem Punkte $2_i\, 2_{03}$.

Die Gesamtheit der Heftgeraden bildet die zu F gehörige Heftfläche Φ; sie ist längs der Heftkurve C_H an F geheftet.

Die Heftgerade ergibt sich auch durch folgende Konstruktion: drei Erzeugenden von F haben eine Transversale, das ist diejenige Gerade, die alle drei Erzeugenden trifft. Lässt man die drei Erzeugenden zusammenrücken, so wird ihre Transversale zur Heftgeraden (38). Aus (37) ist dann abzulesen, dass die Gerade m_{ik} dem durch die Erzeugende und durch die Heftgerade bestimmten Strahlenbüschel angehört.

§ 5.

Vier aufeinanderfolgende Tangentialräume bestimmen im Allgemeinen einen Punkt M auf der durch (37) gegebenen Geraden m_{ik} mit der Gleichung

$$M \ldots (u' M'_{02} M'_{03},\ M'_{13} + M'_{04},\ M'_{23} + 2M'_{14} + M'_{05}) = 0. \quad . \quad (39)$$

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1) Bei algebraischen Regelflächen F im R_4 wird diese Heftgerade von Italienischen Geometern „Transversale“ genannt. Vgl. den Artikel III C 7 (Mehrdimensionale Raume) von C. SEGRE, No. 31, p. 931 (1912), wo auf Arbeiten von M. MORALE (1901) und G. CALDARERA (1896) verwiesen wird.

Ersetzt man hier M'_{02} durch $(0^2\,2^2)$, so lässt sich dies schreiben als

$$\begin{vmatrix} 0 & 0_{13} & 0_{23}+2\,0_{14} & (0\,u') \\ 0 & 0_{13} & 0_{23}+2\,0_{14} & (0\,u') \\ 2_{03} & 2_{13}+2_{04} & 2_{23}+2.2_{14}+2_{05} & (2\,u') \\ 2_{03} & 2_{13}+2_{04} & 2_{23}+2.2_{14}+2_{05} & (2\,u') \end{vmatrix} = 0,$$

also nach (1) auch:

$$\begin{vmatrix} 0 & -\tfrac{3}{4}0_{22} & -3.0_{23} & (0\,u') \\ 0 & -\tfrac{3}{4}0_{22} & -3.0_{23} & (0\,u') \\ 2_{03} & -3.2_{13} & -3.2_{23}+\tfrac{3}{5}2_{05} & (2\,u') \\ 2_{03} & -3.2_{13} & -3.2_{23}+\tfrac{3}{5}2_{05} & (2\,u') \end{vmatrix} = 0.$$

Entwickeln wir dies, so entsteht:

$$\tfrac{3}{4}(0\,u')\,0_{22}\,.\,(3\,.\,2_{03,23}-\tfrac{3}{5}2_{03,05})-\tfrac{9}{4}(2\,u')\,2_{03}\,.\,0_{22,23}-9\,(0\,u')\,0_{23}\,.\,2_{03,13}=0 \quad . \ (40)$$

Im Folgenden gebrauchen wir die Abkürzungen (Vgl. Gleichung (17)):

$$Q=0_{13,23}, \quad R=0_{13,33}, \quad S=0_{23,33}, \quad T=2_{04,33} \quad . \ . \ . \quad (41)$$

Für (40) haben wir dann:

$$0_{22,23}=-\tfrac{4}{3}0_{13,23}=-\tfrac{4}{3}Q$$

$$2_{03,13}=-0_{23,13}-3_{02,13}=Q+3_{11,13}=Q-2.1_{13,13}=Q$$

$$2_{03,23}=-3_{02,23}=+3_{11,23}=-\tfrac{1}{2}2_{11,33}=1_{12,33}=-\tfrac{1}{3}1_{03,33}=+\tfrac{1}{3}0_{13,33}=\tfrac{1}{3}R.$$

Bei diesen Umformungen ist von den Gleichungen (1) und den Symmetrieeigenschaften der Ausdrücke $h_{ik,rs}$ Gebrauch gemacht.

Statt (40) erhalten wir so für den Punkt M, wenn wir noch die Gleichung (21) des Heftpunktes H benutzen:

$$(M\,u')=(H\,u')\,.\,(R+\tfrac{3}{5}2_{05,03})+4\,Q\,[(2\,u')\,2_{03}-3\,.\,(0\,u')\,0_{23}]=0 \quad . \quad (42)$$

Die Kurve, die der Punkt M für veränderliche Parameter t beschreibt, nennen wir C_M. Die Gerade m_{ik} von (37) ist dann die Tangente, die Heftebene (32) ist die Schmiegebene und der Tangentialraum M'_{02} ist der Schmiegraum von C_M im Punkte M.

Wenn fünf aufeinanderfolgende Tangentialräume M'_{02} durch denselben Punkt gehen, so muss der Punkt M von (42) noch dem R_3 mit den Koordinaten

$$v'=M'_{33}+3\,M'_{24}+3\,M'_{15}+M'_{06}$$

angehören. Dies ergibt die Invariante $U=(v'M)$ oder, wenn wir in M noch H durch $0_i\, 0_{22}$ ausdrücken:

$$\left.\begin{aligned} U=(R+\tfrac{3}{5}\,2_{05,03})\,(0_{22,33}+3\,.\,0_{22,24}+3\,.\,0_{22,15})+ \\ +4\,Q\,(2_{03,33}+3\,.\,2_{03,24}+3\,.\,2_{03,15}+2_{03,06})- \\ -12\,Q\,(0_{23,33}+3\,.\,0_{23,24}+3\,.\,0_{23,15}) \end{aligned}\right\} . \quad (43)$$

$U\equiv 0$ ist dann die Bedingung dafür, dass die Klassenkurve der M'_{02} konisch ist, d.h. dass alle Räume M'_{02} durch einen festen Punkt $C_M=M$ gehen.

§ 6.

In der Heftebene m_{ik} haben wir ein durch die Erzeugende 0_{ik} und die zugehörige Heftgerade α_{ik} bestimmtes Strahlenbüschel

$$\vartheta_{ik}=\alpha_{ik}+\lambda\,.\,0_{ik}=(0\,2)_{ik}\,0_{22}\,2_{03}+\lambda\,.\,0_{ik}\,. \quad . \quad . \quad . \quad (44)$$

Für welche λ wird die Regelfläche der ϑ_{ik} abwickelbar? Notwendig und hinreichend hierzu ist nach § 3 das Verschwinden der Kovariante

$$M'_{02}\,(\vartheta)=-M'_{11}\,(\vartheta)=-(x\vartheta_1{}^2\,\omega_1{}^2) \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Dabei ist ω mit ϑ äquivalent und für $(\vartheta_1)_{ik}$ erhalten wir nach (44)

$$(\vartheta_1)_{ik}=(\alpha_1)_{ik}+\lambda\,.\,1_{ik}+\lambda'\,.\,0_{ik}\,. \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Es gilt also zuerst $(\alpha_1)_{ik}$ zu berechnen. Aus (38) finden wir durch Differentiation

$$\left.\begin{aligned} (\alpha_1)_{ik}=(1\,2)_{ik}\,1_{22}\,2_{03}+(03)_{ik}\,0_{22}\,3_{03}+ \\ +2\,.\,(0\,2)_{ik}\,0_{23}\,2_{03}+(0\,2)_{ik}\,0_{22}\,2_{13}+(0\,2)_{ik}\,0_{22}\,2_{04} \end{aligned}\right\} . \quad (47)$$

Hier machen wir beim ersten Term zuerst Gebrauch von der zyklischen Symmetrie bei 1_{22} ($\dot{2}$ ist mit 2 äquivalent):

$$(1\,2)_{ik}\,1_{22}\,2_{03}=-2\,.\,(\dot{2}\,2)_{ik}\,\dot{2}_{12}\,2_{03}.$$

In $\dot{2}_{12}$ ersetzen wir 12 nach (1) durch $-\tfrac{1}{3}\,03$ und erhalten

$$(12)_{ik}\,1_{22}\,2_{03}=+\tfrac{2}{3}\,(\dot{2}2)_{ik}\,\dot{2}_{03}\,2_{03}=0.$$

Beim zweiten Term von (47) berücksichtigen wir die zyklische Symmetrie von 3_{03}:

$$(03)_{ik}\,0_{22}\,3_{03}=-\tfrac{1}{2}\,(0\dot{0})_{ik}\,0_{22}\,\dot{0}_{33},$$

also wegen

$$(0\dot{0}\,\pi^3)\,(0\,u')\,(\dot{0}\,v')=\tfrac{1}{2}\,(0^2\,\pi^3)\,(\dot{0}\,u')\,(\dot{0}\,v'):$$

$$(03)_{ik}\,0_{22}\,3_{03}=-\tfrac{1}{2}\,0_{ik}\,.\,0_{22,33}.$$

In $0_{22,33}$ drücken wir schliesslich 22 nach (1) durch 13 und 04 aus und bekommen nach (41)

$$(03)_{ik}\, 0_{22}\, 3_{03} = + \tfrac{2}{3} R \,.\, 0_{ik}.$$

Beim letzten Term von (47) drücken wir 04 nach (1) aus und haben schliesslich:

$$(\alpha_1)_{ik} = \tfrac{2}{3} R \,.\, 0_{ik} + 2 \,.\, (02)_{ik}\, 0_{23}\, 2_{03} - 3 \,.\, (02)_{ik}\, 0_{22}\, 2_{13}. \quad . \quad . \quad (48)$$

Hiernach wird

$$(\vartheta_1)_{ik} = (\tfrac{2}{3} R + \lambda') \,.\, 0_{ik} + \lambda \,.\, 1_{ik} + 2 \,.\, (02)_{ik}\, 0_{23}\, 2_{03} - 3 \,.\, (02)_{ik}\, 0_{22}\, 2_{13} \quad (49)$$

und damit findet man

$$\left.\begin{array}{l} \tfrac{1}{4} (\vartheta^2 \omega^2 x) = \tfrac{1}{4} \lambda^2 \,.\, (1^2 \,\dot{1}^2 x) + 2 \lambda \,.\, (021^2 x)\, 0_{23}\, 2_{03} - \\ - 3 \lambda \,.\, (021^2 x)\, 0_{23}\, 2_{03} - 12 \,.\, (0\dot{2}\dot{0}\dot{2} x)\, 0_{23}\, 2_{03}\, \dot{0}_{22}\, \dot{2}_{13}. \end{array}\right\} \quad . \quad . \quad . \quad (50)$$

Der erste Term ist hier

$$\tfrac{1}{4} \lambda^2 \,.\, (1^2 \,\dot{1}^2 x) = \tfrac{1}{4} \lambda^2 \,.\, x_{11} = \tfrac{1}{4} \lambda^2 \,.\, (M'_{11}\, x) = - \tfrac{1}{4} \lambda^2 \,.\, (x M'_{02}).$$

Beim zweiten Term ersetzen wir zuerst 03 nach (1) durch -3.12 und machen dann von der zyklischen Symmetrie bezüglich 2_{12} Gebrauch:

$$2 \lambda \,.\, (021^2 x)\, 0_{23}\, 2_{03} = - 6 \lambda \,.\, (021^2 x)\, 0_{23}\, 2_{12} = + 3 \lambda \,.\, (0\dot{1}1^2 x)\, 0_{23}\, \dot{1}_{22}.$$

Hier bringen wir die beiden Reihen $\dot{1}$ in den Klammerfaktor und haben dann wegen

$$0_{11,23} = 0 \quad , \quad 0_{22,23} = - \tfrac{4}{3}\, 0_{13,23} = - \tfrac{4}{3} Q:$$
$$2 \lambda \,.\, (021^2 x)\, 0_{23}\, 2_{03} = \lambda Q \,.\, x_{11}.$$

Analoge Umformung bei den zwei letzten Termen in (50) führen dann zur Formel

$$(\vartheta^2 \omega^2 x) = (\lambda^2 - 16 Q^2) \,.\, (x M'_{11}), \quad . \quad . \quad . \quad . \quad . \quad (51a)$$

was wir auch so schreiben können

$$M'_{02} (\alpha_{ik} + \lambda \,.\, 0_{ik}) = (\lambda^2 - 16 Q^2) \,.\, M'_{02}. \quad . \quad . \quad . \quad . \quad (51b)$$

Ist $\lambda = 0$, so gibt dies

$$M'_{02} (\alpha_{ik}) = - 16 Q^2 \,.\, M'_{02}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

d.h. der zur Heftgeraden gehörige Tangentialraum an die Heftfläche fällt mit dem Tangentialraum M'_{02} zusammen wenn die Invariante $Q \neq 0$ ist. Für $Q \equiv 0$ ist also die Heftfläche abwickelbar.

Aus (51) erhalten wir im Büschel $a_{ik}+\lambda . 0_{ik}$ zwei Geraden, die abwickelbare Flächen erzeugen: $\lambda = \pm 4Q$. Für $\lambda = +4Q$ ergibt sich die durch (37) gegebene Tangente m_{ik} an die Kurve C_M. Der zweite Wert $\lambda = -4Q$ führt zur Geraden

$$\tfrac{3}{2}\, h_{ik} = a_{ik} - 4\,Q\,.\,0_{ik}. \qquad (53)$$

§ 7.

Der zur Erzeugenden 0_{ik} gehörige Heftpunkt ist gegeben durch

$$H = (H u') = (0\, u')\, 0_{22} \qquad \text{(vgl. Gleichung (21))}.$$

Wir setzen wieder

$$H_1 = \frac{dH}{dt}, \quad H_2 = \frac{d^2 H}{dt^2}, \ldots .$$

und erhalten

$$(H_1\, u') = 2\,.\,(0\, u')\, 0_{23} + (1\, u')\, 1_{22}. \qquad (54)$$

$$(H_2\, u') = 4\,.\,(1\, u')\, 1_{23} + 2\,.\,(0\, u')\, 0_{33} + 2\,.\,(0\, u')\, 0_{24} \qquad (55)$$

Bei H_1 kann man den letzten Term umformen:

$$(1\, u')\, 1_{22} = -\, 2\,.\,(2\, u')\, 2_{12} = +\, \tfrac{2}{3}\,(2\, u')\, 2_{03},$$

sodass wir also auch haben

$$(H_1\, u') = \tfrac{2}{3}\,[3\,.\,(0\, u')\, 0_{23} + (2\, u')\, 2_{03}]. \qquad (56)$$

Mit diesem Ausdrucke ergibt sich für die Tangente h_{ik} im Heftpunkte H an die Heftkurve C_H:

$$h_{ik} = (H H_1)_{ik} = \tfrac{2}{3}\,[3\,.\,(0\dot{0})_{ik}\, 0_{22}\, \dot{0}_{23} + (02)_{ik}\, 0_{22}\, 2_{03}]$$

und dies führt auf

$$h_{ik} = \tfrac{2}{3}\,(a_{ik} - 4\,Q\,.\,0_{ik}) = (H H_1)_{ik}, \qquad (57)$$

also auf (53). Wir sehen also, dass die zweite Gerade des Büschels $a_{ik}+\lambda . 0_{ik}$, die eine abwickelbare Regelfläche erzeugt, die Tangente h_{ik} an die Heftkurve C_H ist.

Für die Heftgerade a_{ik} ergibt sich dann die Darstellung

$$a_{ik} = \tfrac{3}{2}\,(H H_1)_{ik} + 4\,Q\,.\,0_{ik} \qquad (58)$$

und m_{ik} von (37) kann geschrieben werden als

$$m_{ik} = \tfrac{3}{2}\,(H H_1)_{ik} + 8\,Q\,.\,0_{ik} = a_{ik} + 4\,Q\,.\,0_{ik}. \qquad (59)$$

Für das Büschel $a_{ik}+\lambda . 0_{ik}$ können wir also auch die Darstellung

$$\varphi_{ik}=\tfrac{3}{2}(HH_1)_{ik}+\lambda . 0_{ik}=a_{ik}+(\lambda-4Q) . 0_{ik} \quad . \quad . \quad . \quad (60)$$

wählen, wobei jetzt die Werte $\lambda=4Q$ und $\lambda=8Q$ die Geraden a_{ik} und m_{ik} geben. Dieses φ_{ik} von (60) entsteht jetzt aus dem ϑ_{ik} von (44), wenn wir λ durch $\lambda-4Q$ ersetzen. Nach (51*b*) haben wir jetzt:

$$M'_{02}(\varphi)=\lambda(\lambda-8Q) . M'_{02}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60a)$$

Wir berechnen jetzt $H(\varphi)$, d.h. den zur Geraden φ_{ik} gehörigen Heftpunkt

$$H(\varphi)=(\varphi u')\,\varphi_{\varphi_2\varphi_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

wobei $\varphi_2 . \varphi_2$ aus

$$M'_{22}(\varphi_{ik})=(\varphi_2{}^2\,\psi_2{}^2\,x)=4 . \Sigma(\varphi_2)_{12}(\varphi_2)_{34}\,x_5$$

zu entnehmen ist.

Wir haben durch Differentiation von (60):

$$(\varphi_1)_{ik}=\tfrac{3}{2}(HH_2)_{ik}+\lambda . 1_{ik}+\lambda' . 0_{ik} \quad . \quad . \quad . \quad . \quad . \quad (62)$$

$$(\varphi_2)_{ik}=\tfrac{3}{2}(H_1H_2)_{ik}+\tfrac{3}{2}(HH_3)_{ik}+\lambda . 2_{ik}+2\lambda' . 1_{ik}+\lambda'' . 0_{ik}. \quad (63)$$

Aus der letzten dieser Gleichungen ergibt sich

$$\left.\begin{aligned} M'_{22}(\varphi_{ik}) = \lambda^2 . x_{22}+4\lambda'^2 . x_{11}+18(HH_1H_2H_3x)+6\lambda . (2^2H_1H_2x)+\\ +12\lambda' . (1^2H_1H_2x)+6\lambda'' . (0^2H_1H_2x)+6\lambda . (2^2HH_3x)+\\ +12\lambda' . (1^2HH_3x)+4\lambda\lambda' . x_{12}+2\lambda\lambda'' . x_{02}.\end{aligned}\right\} (64)$$

Hier gibt der dritte Term rechts

$$(H'x)=(HH_1H_2H_3x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

den Schmiegraum in H an die Heftkurve C_H. Die übrigen Terme mit Klammerfaktoren kann man auf die $x_{ik}=(xM'_{ik})$ umformen. So ist z.B. nach (54):

$$(1^2H_1H_2x)=2(01^2H_2x)\,0_{23}+(1^2\,\mathrm{i}\,H_2x)\,\mathrm{i}_{22}.$$

Hier bringen wir im ersten Gliede rechts die beiden Reihen 0 in den Klammerfaktor:

$$2(01^2H_2x)\,0_{23}=-2(0^2\,1\,H_2x)\,1_{23},$$

also nach (55):

$$2(01^2H_2x)\,0_{23}=-8(0^2\,1\,\mathrm{i}\,x)\,1_{23}\,\mathrm{i}_{23}=0.$$

Weiter ist

$$(1^2\,\mathrm{i}\,H_2x)\,\mathrm{i}_{22}=\tfrac{1}{4}x_{11}(H_2)_{22}-\tfrac{1}{4}x_{22}(H_2)_{11},$$

sodass wir schliesslich haben

$$(1^2\,H_1\,H_2\,x) = -\tfrac{1}{4}\,x_{02}\,(H_2)_{22} - \tfrac{1}{4}\,x_{22}\,(H_2)_{11}.$$

Analog bei den übrigen Termen von (64). Für einige der hier auftretenden Koeffizienten $(H_i)_{rs}$ findet man leicht nach (54) und (55):

$$\left.\begin{aligned}
&(H)_{0i} = 0,\ (H)_{22} = 0,\ (H)_{23} = 0_{23,22} = \tfrac{4}{3}\,Q,\ (H)_{33} = \tfrac{4}{3}\,R\\
&(H_1)_{02} = 0,\ (H_1)_{03} = 0,\ (H_1)_{22} = -\tfrac{8}{3}\,Q,\ (H_1)_{04} = \tfrac{8}{3}\,Q,\ (H_1)_{23} = -\tfrac{2}{9}\,R\\
&(H_2)_{02} = 0,\ (H_2)_{03} = 4.1_{03,23} = -4\,Q\\
&(H_3)_{02} = 4.2_{02,23} = \tfrac{8}{3}\,Q,\ (H_3)_{03} = \tfrac{1}{3}\,R - 6\,Q'.
\end{aligned}\right\} \quad (66)$$

Man erhält nach einiger Rechnung statt (64):

$$M'_{22}\,(\varphi_{ik}) = 18\,(HH_1\,H_2\,H_3\,x) + B_1\,x_{02} + B_2\,x_{03} + B_3\,x_{04} + B_4\,x_{22}, \quad (67)$$

wobei die Koeffizienten B gegeben sind durch die Gleichungen

$$\left.\begin{aligned}
&B_1 = -4\,\lambda'^2 + 2\,\lambda\lambda'' - 3\,\lambda'\,(H_2)_{22} - 4\,\lambda'\,(H_3)_{03} + \tfrac{16}{3}\,\lambda''\,Q + 3\,\lambda\,(H_3)_{22} + 6\,\lambda\,.\,2_{04,23}\\
&B_2 = -\tfrac{4}{3}\,\lambda\lambda' + \tfrac{32}{3}\,\lambda'\,Q - \lambda\,(H_2)_{22}\\
&B_3 = -4\,\lambda\,Q\\
&B_4 = \lambda^2 - 12\,\lambda\,Q.
\end{aligned}\right\} \quad (68)$$

Nennen wir die rechte Seite von (67) $(w'\,x)$, dann wird nach (61) und (60)

$$\begin{aligned}
H\,(\varphi) &= \Sigma\,(u'\,w')_{ik}\,\varphi_{ik} = \tfrac{3}{2}\,\Sigma\,(u'\,w')_{ik}\,(HH_1)_{ik} + \lambda\,(0u')\,(0w') =\\
&= \tfrac{3}{2}\,[(u'H)(w'H_1) - (u'H_1)(w'H)] - \lambda\,[18\,(0HH_1\,H_2H_3)(0u') + B_4\,0_{22}\,(0u')].
\end{aligned}$$

Hier findet man weiter nach (67) und (66)

$$\begin{aligned}
&(w'\,H) = 0\\
&(w'\,H_1) = \tfrac{8}{3}\,Q\,(B_3 - B_4) = \tfrac{8}{3}\,\lambda\,Q\,(8\,Q - \lambda)
\end{aligned}$$

und bei $(0\,HH_1\,H_2\,H_3)\,(0\,u')$ erhält man nach Umformung und nach (66)

$$2\,(0HH_1\,H_2\,H_3)\,(0u') = (0^2\,H_1\,H_2\,H_3)\,(Hu') = -\tfrac{32}{9}\,Q^2\,.\,(Hu'). \quad (69)$$

Setzt man dies alles in $H\,(\varphi)$ ein, so kommt schliesslich

$$H\,(\varphi) = H\,(\tfrac{3}{2}\,(HH_1)_{ik} + \lambda\,.\,0_{ik}) = \lambda\,(\lambda - 8\,Q)^2\,.\,(Hu') \quad . \quad . \quad (70)$$

$H\,(\varphi)$ verschwindet also für $\lambda = 0$ (Gerade h_{ik}) und für $\lambda = 8Q$ (Gerade m_{ik}, vgl. Gleichung (59)). Für die Heftgerade a_{ik} ist nach (58) $\lambda = 4Q$ zu setzen, d.h. es ist

$$H\,(a) = 64\,Q^3\,.\,(Hu') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

oder: der Heftpunkt der Heftgeraden a_{ik} bezüglich der Heftfläche fällt, falls $Q \neq 0$, mit dem Heftpunkt H zusammen.

§ 8.

Wenn wir in

$$Q = 0_{13,23}$$

nach (1) das Paar 13 durch 22 ausdrücken und 23 durch die Ableitung $\frac{1}{2}(22)_1$ ersetzen, so lässt sich die Invariante Q so schreiben:

$$Q = -\tfrac{3}{4}\, 0_{22,23} = \tfrac{3}{8}\, 0_{(22)_1}\, 0_{22} = \tfrac{3}{8}\, (H)_{(22)_1}. \quad . \quad . \quad . \quad . \quad (72)$$

Auf Grund dieser Darstellung lässt sich $Q(\varphi)$ einfach berechnen, wobei die Gerade φ_{ik} durch (60) gegeben ist. Es wird

$$Q(\varphi) = \tfrac{3}{8}\, [H(\varphi)]_{[M'_{22}(\varphi)]_1},$$

also nach (70) und wenn wir wieder die rechte Seite von (67) mit $(w'x)$ bezeichnen:

$$Q(\varphi) = \tfrac{3}{8}\, \lambda\, (\lambda - 8\, Q)^2 \,.\, (H w_1').$$

Da $(H w') = 0$ ist, folgt durch Differentiation

$$(H_1 w') + (H w_1') = 0 \;\;,\;\; (H w_1') = -(H_1 w'),$$

also, da wir oben schon $(H_1 w')$ berechnet haben,

$$(H w_1') = -\tfrac{8}{3}\, Q\, \lambda\, (8\, Q - \lambda). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

Somit wird

$$Q(\varphi) = -Q \,.\, \lambda^2\, (8\, Q - \lambda)^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

Für $\lambda = 4Q$ gibt dies für die Heftgerade α_{ik}:

$$Q(\alpha) = -4^5 \,.\, Q^6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

Weiters berechnen wir die zu φ_{ik} gehörige Heftgerade $\alpha(\varphi)$. Nach (58) ist

$$\alpha(\varphi) = \tfrac{3}{2}\, (H(\varphi)\, H_1(\varphi))_{ik} + 4\, Q(\varphi) \,.\, \varphi_{ik}.$$

Aus (70) finden wir

$$(H(\varphi)\, H_1(\varphi))_{ik} = \lambda^2\, (8\, Q - \lambda)^4 \,.\, (H H_1)_{ik};$$

also wird nach (74) und (60):

$$\alpha(\varphi) = \tfrac{3}{2}\, \lambda^2 (8\, Q - \lambda)^4 \,.\, (H H_1)_{ik} - 4\, Q\, \lambda^2\, (8\, Q - \lambda)^3 \,.\, [\tfrac{3}{2}(H H_1)_{ik} + \lambda \,.\, 0_{ik}]$$

$$\alpha(\varphi) = \lambda^2 (8\, Q - \lambda)^3 \,.\, [\tfrac{3}{2}(4\, Q - \lambda) \,.\, (H H_1)_{ik} - 4\, Q\, \lambda \,.\, 0_{ik}] \quad . \quad . \quad (76)$$

Setzen wir hier $\lambda = 4Q$, so ergibt sich

$$\alpha_{ik}(\alpha) = (-4\, Q)^7 \,.\, 0_{ik} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

Für $Q \neq 0$ ist also die Heftfläche der Heftfläche mit der ursprünglichen Regelfläche F identisch, was geometrisch zu erwarten war.

Astronomy. — *Spontaneous development of a gaseous disc revolving round the sun into rings and planets.* II. By H. P. BERLAGE Jr., Research Associate Roy. Magn. and Meteorological Observatory, Batavia.

(Communicated at the meeting of March 30, 1940.)

Whether the nebula will really start producing rings depends on the condition that its potential energy decreases during the process. This condition requires that the left side of (42) is positive. Controlling this for the values of x, φ, ψ, γ and δ already found, we have to be extremely careful. We know that these values have to be slightly corrected in order to satisfy the 3 equations (40) (41) (42) in their unapproximated form. The application of this correction, however insignificant, has now become essential, because otherwise we should perhaps overlook that the left side of (42) which is to be positive, is identically $= 0$ when we take account in the variations of small quantities of the first order of magnitude only.

Indeed, as the nebula is supposed to start on its new course from a state of instable equilibrium the variation applied leaves $\triangle U$ and $\triangle V$ both $= 0$ up to first order small quantities. The variation of U and V in second order small quantities will decide whether U and V are decreasing or increasing.

The only way out of insurmountable complications is then to leave the general track and to start the calculation with a value of x which has by trial been found to agree with the actual state of the planetary system. We are then able to show that the nebula was inclined to follow the suggested course.

A very good approximation to the actual conditions is obtained for

$$a = 7.4 \times 10^{-7},\ b = 14.8 \times 10^{-7},\ x = 0.5 \quad . \quad . \quad . \quad . \quad (48)$$

We find

$$\frac{\gamma}{\delta} = -0.441 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Let us take as the one independent variable the important "amplitude" $\varkappa$ of the density fluctuation, or

$$\varkappa^2 = \gamma^2 + \delta^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

We then get successively

$$\gamma = -0.404\,\varkappa,\ \delta = 0.916\,\varkappa,\ \psi = -0.00533\,\varkappa a,\ \varphi = -0.0408\,\varkappa. \quad (51)$$

How is the structure of the planetary system that would be created in this way? The density maxima are along circles satisfying (21) or

$$\sqrt{r_n} - \sqrt{r_{n-1}} = \frac{2\pi}{b} = 4.24 \times 10^6 \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

If r_1 is the radius of the smallest ring

$$tg \frac{7.4 \times 10^{-7} \times r_1^{\frac{1}{2}}}{0.5} = 0.441 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

or

$$r_1^{\frac{1}{2}} = 0.28 \times 10^6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

Hence

$$r_n^{\frac{1}{2}} = (0.28 + 4.24\, n) \times 10^6 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

and

$$r_n = (0.28 + 4.24\, n)^2 \times 10^{12}\ \text{cm} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

When writing this series out we obtain

0.078 20.5 77.2 169 297 462 660 889 etc. $\times$ 10^{12} cm

or in astronomical units

0.005 1.37 5.15 11.3 19.8 30.8 43.9 60.0 etc.

Numbering the planets accordingly

I II III IV V VI VII VIII

suitable distances from the sun are obtained for

	actual mean distance from the sun in A.U.
III = Jupiter	5.2
IV = Saturn	9.5
V = Uranus	19.2
VI = Neptune	30.4
VII = Pluto	40

Is it by chance that r_1 nearly corresponds with the sun's radius $r_0 = 7 \times 10^{10}$ cm? If not, we could have found x if we had put in (47) for r_m the solar radius r_0. Is it not very likely that the only actually existing "boundary condition", a density optimum along the equator of the solar globe has fixed the value of x? If this be so, with given a and ϱ_0 the course of evolution could have been predicted quantitatively.

Now, let us calculate the masses of the planets. This can be done with a planimeter, when we have drawn the curve

$$r^{\frac{5}{2}}\, e^{-7.4 \times 10^{-7}\, r^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

by measuring the successive surfaces between the axis of abscissae, the curve and ordinates through the minima of

$$-\gamma \sin b\, r^{\frac{1}{2}} + \delta \cos b\, r^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

If then we make the sum total of the calculated masses of Jupiter, Saturn, Uranus, Neptune and Pluto equal to the sum total of the actual masses, we get

TABLE I.

Planet	Distance in A.U.	Mass		
		in grams $\times 10^{29}$	earth $= 1$	actual
I	0.005	0.09	1.5	
II	1.37	7.19	122	
Jupiter	5.15	15 99	271	318
Saturn	11.3	7.83	133	95
Uranus	19.8	2.09	35.4	14.6
Neptune	30.8	0.34	5.9	17.3
Pluto	43.9	0.05	0.8	0.8
Total		33.58	569.6	

The agreement between the computed and actual masses is promising, but it is difficult to avoid the conclusion *that the planetary system was not created in one act.* Ring I will have been united with the sun and ring II surely never gave birth to that big planet with a mass equal to 122 times the mass of the Earth. In its place appeared Mercury, Venus, Earth, Mars and the planetoids, most probably in a second act of creation. We will come back to this later.

We concluded that a second approximation of $\triangle U$ and $\triangle V$ only could decide which of both is positive or negative. Hence we have to develop (15) into

$$\left.\begin{array}{l} \varrho_e + \triangle \varrho_e = \varrho_0\, e^{-ar^{\frac{1}{2}}}\,(1+\varphi)\left(1-\psi r^{\frac{1}{2}}+\frac{\psi^2}{2}r\right) \\ \{1-2\gamma \sin b r^{\frac{1}{2}}+2\delta \cos br^{\frac{1}{2}}+(\gamma^2+\delta^2)+(\delta^2-\gamma^2)\cos 2\, br^{\frac{1}{2}}+2\gamma\delta \sin 2\, br^{\frac{1}{2}}\} \end{array}\right\} \quad (59)$$

or

$$\left.\begin{array}{l} \dfrac{\triangle \varrho_e}{\varrho_e} = \varphi+(\gamma^2+\delta^2)-\psi(1+\varphi)r^{\frac{1}{2}}+(1+\varphi)(-2\gamma \sin br^{\frac{1}{2}}+2\delta \cos br^{\frac{1}{2}})+ \\ +\dfrac{\psi^2}{2}r-\psi r^{\frac{1}{2}}(-2\gamma \sin br^{\frac{1}{2}}+2\delta \cos br^{\frac{1}{2}})+\}(\delta^2-\gamma^2)\cos 2\, br^{\frac{1}{2}}+2\gamma\delta \sin 2\, br^{\frac{1}{2}}\} \end{array}\right\} \quad (60)$$

Changing in our previous equations (40) (41) (42)

$$\begin{array}{ll} \varphi & \text{into } \varphi+(\gamma^2+\delta^2) \\ \psi & \psi(1+\varphi) \\ \gamma & \gamma(1+\varphi) \\ \delta & \delta(1+\varphi) \end{array}$$

we get the new equations as far as first order small quantities. Evidently, up to these quantities the solution can be obtained in the same way. We shall get again $\triangle U = 0$ and $\triangle V = 0$, but now with values of φ, ψ, γ, δ slightly different from the former solution. The second order quantities decide about the sign of $\triangle U$ and $\triangle V$. The natural course is indicated by

$$\triangle U > 0$$

and now this changes into the condition

$$\left.\begin{array}{r}\dfrac{4\,!}{\frac{1}{2}a^5}\left[15\left(\dfrac{\psi}{a}\right)^2-12\left(\dfrac{\psi}{a}\right)x^6(x^2+1)^{-6}\left[-\gamma\{-6x^5+21x^3-6x\}+\right.\right.\\ \left.+\delta\{x^6-15x^4+15x^2-1\}\right]+\frac{1}{32}\left(\tfrac{1}{2}x\right)^5\left(\tfrac{1}{4}x^2+1\right)^{-5}\\ \left.\left[-\gamma\delta\{-20x^4+160x^2-64\}+(\delta^2-\gamma^2)\{x^5-40x^3+80x\}\right]\right]>0\end{array}\right\} . \quad (62)$$

Substituting $x=0.5$ it reduces to

$$0.002015\,\delta^2>0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

or

$$0.00168\,\varkappa^2>0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

The left side being positive the condition is satisfied and we have proved that the gaseous disc will show spontaneously a tendency to concentration into concentric rings.

If U_0 is the original kinetic energy of the system

$$\triangle U=0.00168\,\varkappa^2\,U_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

We are now in the position to calculate

$$\theta_0 \text{ and } \theta_e, \quad U_0 \text{ and } U_e$$

the moment of momentum and kinetic energy of the planetary system at the beginning and at the end of its evolution from the disc.

If m be the mass of a planet and r its final distance from the sun, we get the following table.

TABLE II.

Planet	m	$r^{\frac{1}{2}}$	r	$m\,r^{\frac{1}{2}}$	$\dfrac{m}{r}$
I	0.09×10^{29}	0.28×10^{6}	0.0784×10^{12}	0.0×10^{35}	1.15×10^{16}
II	7.19	4.52	20.5	32.5	3.51
Jupiter	15.99	8.76	77.2	140	2.08
Saturn	7.83	13.00	169	102	0.46
Uranus	2.09	17.24	297	36.2	0.07
Neptune	0.34	21.48	462	7.3	0.01
Pluto	0.05	25.72	660	1.3	0.00
Σ	33.58×10^{29}			319.3×10^{35}	7.28×10^{16}

But

$$\theta_e=(f\,M)^{\frac{1}{2}}\,\Sigma\,m\,r^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

or with $f=6.67\times10^{-8}$ and $M=2.00\times10^{33}$

$$\theta_e=3.69\times10^{50} \quad \text{c. g. s.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

whereas

$$U_e = \tfrac{1}{2} f M \Sigma \frac{m}{r} \quad \ldots \ldots \ldots \quad (68)$$

or

$$U_e = 4.84 \times 10^{42} \quad \text{c. g. s.} \quad \ldots \ldots \quad (69)$$

The total mass, total moment of momentum and kinetic energy of the disc are found by

$$m_0 = (2\pi)^{\frac{3}{2}} \left(\frac{R\,T}{f\,M}\right)^{\frac{1}{2}} \varrho_0 \frac{6\,!}{\frac{1}{2}a^7} \quad \ldots \ldots \quad (70)$$

$$\theta_0 = (2\pi)^{\frac{3}{2}} (R\,T)^{\frac{1}{2}} \varrho_0 \frac{7\,!}{\frac{1}{2}a^8} \quad \ldots \ldots \quad (71)$$

$$U_0 = \tfrac{1}{2} (2\pi)^{\frac{3}{2}} (f\,M\,R\,T)^{\frac{1}{2}} \varrho_0 \frac{4\,!}{\frac{1}{2}a^5} \quad \ldots \ldots \quad (72)$$

with quite sufficient approximation. Substituting provisionally

$$(R\,T)^{\frac{1}{2}} \varrho_0 = y \quad \ldots \ldots \ldots \quad (73)$$

we obtain

$$(2\pi)^{\frac{3}{2}} \frac{y}{(f\,M)^{\frac{1}{2}}} \frac{6\,!}{\frac{1}{2}a^7} = 335.9 \times 10^{28} \quad \ldots \ldots \quad (74)$$

or

$$y = 2.08 \times 10^{-4} \quad \ldots \ldots \ldots \quad (75)$$

We then get

$$\theta_0 = 3.68 \times 10^{50} \quad \text{c. g. s.} \quad \ldots \ldots \quad (76)$$

and

$$U_0 = 4.10 \times 10^{42} \quad \text{c. g. s.} \quad \ldots \ldots \quad (77)$$

Comparing (67) and (76) we can confirm that the moment of momentum of the system has not changed from the origin, whereas (69) shows that the kinetic energy has increased. The total increase is represented by

$$\triangle\, U = 0.18\, U_0 \quad \ldots \ldots \ldots \quad (78)$$

It is of interest to compare

$$\triangle\, U = 0.00168\, \varkappa^2\, U_0 \quad \ldots \ldots \quad (65)$$

Although the approximations used in our calculations allowed it to be carried through only in cases when $\varkappa \ll 1$, we are now permitted to conclude that $\varkappa$ might at most have reached a value comparable with 10.

The value of $\varkappa$ actually reached depended on the limit of stability of

the rings. In previous papers I have shown that a critical limit in the motion of our rotating nebula is reached, when

$$\frac{fM}{RT}+\frac{d}{dr}\left(r^3\frac{d\,lg\,\varrho_e}{dr}\right)=0\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (79)$$

This limit is fixed by the possibility of adiabatic radial displacements of masselements within the nebula. It is the limit where laminar motion is replaced by turbulent motion. The left side of (79) can never be negative.

Introducing in (79)

$$\varrho_e=\varrho_0\, e^{-ar^{\frac{1}{2}}-2\gamma\sin br^{\frac{1}{2}}+2\delta\cos br^{\frac{1}{2}}}$$

we get the following condition of stability

$$\frac{fM}{RT}-\frac{5}{2}br^{\frac{3}{2}}\left[\frac{a}{2b}+\delta\sin br^{\frac{1}{2}}+\gamma\cos br^{\frac{1}{2}}\right]-\frac{b^2}{2}r^2[-\gamma\sin br^{\frac{1}{2}}+\delta\cos br^{\frac{1}{2}}]>0 \quad (80)$$

Trying it out quantitatively, let us substitute

$$a=7.4\times10^{-7} \text{ and } b=14.8\times10^{-7}.$$

As we have to try out values of $\varkappa$ many times greater than 1, we can as well neglect the term $\frac{a}{2b}=\frac{1}{4}$ between the first brackets. After having done this, both terms between brackets reach the absolute value of $\varkappa$ at their maximum. Hence the second and third terms on the left side are comparable for

$$r^{\frac{1}{2}}=\frac{5}{b} \text{ or } r=11.4\times10^{12}\,\text{cm}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (81)$$

Beyond this distance, which is smaller than one A.U. the third term is the more dangerous one. We may therefore conclude that, roughly, a ring becomes instable, when $\varkappa$ has increased as far as to satisfy the equation

$$\frac{fM}{RT}-\frac{\varkappa}{2}b^2r^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (82)$$

or when

$$\varkappa=\frac{12.1\times10^{37}}{RT\,r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (83)$$

When this critical value of $\varkappa$ is reached it allows locally small parts of the mass to unite freely by their own attraction. Concentration to small solid particles will start, the formation of meteoric bodies is stimulated. However, the final concentration of these bodies or of a ring, which has remained gaseous, into one or more big bodies, depends on the mean density which the ring has attained. This second critical limit which has to be crossed is ROCHE's limit.

It is not the author's intention to deal in this paper with the formation

of satellites more fully. But here we have to mention it in passing. The rings of Saturn, rotating within ROCHE's limit prove that condensation to meteorites was permitted, whereas it is known that the mean density of some of the inner satellites of Saturn is so low that they must be clouds of solid particles like comets heads.

It may be that every planet passed through the planetesimal state, but certainly those planets did, which in their evolution crossed the first mentioned limit of $\varkappa$ before ROCHE's limit was crossed.

This second limit follows from the condition that the density of the rings is sufficiently high to ascertain their condensation into individual bigger globes. Applying ROCHE's theory this condition is satisfied, when, roughly, the mean density of a ring has reached 14 times the mean density of a central body with the ring at its circumference. To start the concentration, at least the maximum density in any ring has to reach this value. Hence, a ring becomes instable when $\varkappa$ has increased as far as to satisfy the equation

$$\varrho_0 \, e^{-ar^{\frac{1}{2}}+2\varkappa} = 14 \times \frac{M}{\frac{4}{3}\pi r^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

or

$$0.434\,(-ar^{\frac{1}{2}} + 2\varkappa) = {}^{10}log\left(\frac{14}{\varrho_0} \times \frac{M}{\frac{4}{3}\pi r^3}\right) \quad . \quad . \quad . \quad . \quad (85)$$

With

$$\varrho_0 = 2.08 \times 10^{-4}\,(RT)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

we obtain

$$\varkappa = 1.15\,(3.21 \times 10^{-7}\, r^{\frac{1}{2}} + 37.0 + \tfrac{1}{2}\, log\, RT - 3\, log\, r) \quad . \quad . \quad (87)$$

To fix the ideas, let us assume that in our tenuous slightly absorbing original nebula, we have at earthdistance from the sun

$R = 2 \times 10^6$ (a gas with $\mu = 40$), $T = 100^\circ$ (blackbody-temperature $= 280^\circ$)

hence, with our previous assumption (4), anywhere

$$RT = 2 \times 10^8 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

We then get for the density of the nebula at the centre, that is in contact with the sun

$$\varrho_0 = 1.48 \times 10^{-8} \text{ gr cm}^{-3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

and for the two critical limits of $\varkappa$

$$\varkappa_1 = 6.05 \times 10^{29}\, r^{-2}, \quad \varkappa_2 = 1.15\,(3.21 \times 10^{-7}\, r^{\frac{1}{2}} + 41.15 - 3\, log\, r) \quad (90)$$

Writing these values out, the following table is obtained

TABEL III.

	K_1	K_2
II	1440	3.08
Jupiter	101	2.63
Saturn	21.2	3.03
Uranus	6.84	3.76
Neptune	2.83	4.66
Pluto	1.40	5.68

I dare say that this table contains the clue to many problems. However, how the breaking up of the rings and the formation of the embryos of the planets takes place, we can only guess. Nor are we sure of the absolute magnitude of the figures in the table, in consequence of the uncertainty in (88). Taking the values in the table as they are, and remembering that the lower of the two limits of $\varkappa$ is decisive, we have to assume that the generation of the planets proceeded in the following surprising order of succession

Pluto	$\varkappa_1 = 1.40$
Jupiter	$\varkappa_2 = 2.63$
Neptune	$\varkappa_1 = 2.83$
Saturn	$\varkappa_2 = 3.03$
II	$\varkappa_2 = 3.08$
Uranus	$\varkappa_2 = 3.76$

It is of importance to remark that $\varkappa$ remains inferior to 10, a value which, as we have seen, could hardly have been surpassed. Pluto and Neptune reached $\varkappa_1$ before $\varkappa_2$. They were in danger to remain in the form of planetoids. Neptune has evidently escaped this danger, Pluto perhaps not. Its excentric behaviour suggests that there may be other "plutoids". Moreover the next planet beyond Pluto, revolving at a mean distance of 60 A.U. would still possess, according to our theory, appreciable mass, say 0.1 of the mass of the Earth. Hence we have to leave open the possible existence of a peripheral ring of meteorites.

It completes our picture, when we imagine that with $\varkappa = 3$ at the moment of the final breaking up of the rings, the densities reached at the crests are e^{12} or 10^5 times superior to the densities in the near valleys.

Looking away from Pluto, Neptune and Jupiter started condensation. This might perhaps be the reason, why Jupiter and Neptune are heavier and the other planets lighter than they ought to be, according to table I. When trying to grasp what will happen, it appears probable that when a ring starts condensation into one or more separate bodies the other rings will start supplying matter to the ring in progress of disintegration in order to restore the violated state of equilibrium in the nebula. So Jupiter grew partly at the cost of II and Saturn, Neptune at the cost of Uranus. And it becomes more and more probable that at last all these planets came into

existence with the exception of II which lost almost all its matter before succeeding in its own concentration. However, the big planets once created, there came a moment when the inconsiderable rest of the gas inside the orbit of Jupiter, not more than 2 times the earth's mass, was once more reconstructed in discform and rearranged in a series of concentric rings, giving birth to a new planetary system much smaller than the first, but of a comparable character

	mass (earth = 1)
Mercury	0.06
Venus	0.82
Earth	1
Mars	0.11
Planetoids	very small

The author has convinced himself that new values a, b and ϱ_0, suiting reasonably well the masses and distances of these planets can be found. However, he assumes that the solid base now obtained allows him to be short and to leave this point as well as the formation of the satellite systems to future investigation.

It is of interest to get a better idea of the structure of the rings before they condensed to planets. Developing the density ϱ in the neighbourhood of a maximum at radius r_m, we get

$$\varrho = \varrho_0 (1 + \varrho)\, Exp \left[-(\alpha + \psi)\, r^{\frac{1}{2}} - \frac{fM}{2RT} \frac{h^2}{r^3} + 2\varkappa \left\{ 1 - \tfrac{1}{8} b^2 \frac{(r - r_m)^2}{r} \right\} \right] \quad (91)$$

Putting

$$r - r_m = k \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (92)$$

the equation of the curves of equal density in a meridional section through a ring is

$$\frac{fM}{2RT} \frac{h^2}{r^3} + \tfrac{1}{4} \varkappa\, b^2 \frac{k^2}{r} = \text{constant} \quad . \; . \; . \; . \; . \quad (93)$$

For the smaller values of h and k these curves are ellipses. If the major and minor axes are 2α and 2β we have

$$\frac{\beta}{\alpha} = \left(\frac{\varkappa RT}{2fM} \right)^{\frac{1}{2}} br \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (94)$$

This ratio increases with r. Inserting known values we get

$$\frac{\beta}{\alpha} = \varkappa^{\frac{1}{2}} \times 1.28 \times 10^{-15}\, r \quad . \; . \; . \; . \; . \; . \; . \; . \quad (95)$$

To fix the ideas, let us put $\varkappa = 3$ at the moment, when the rings loose their stability. Then

$$\frac{\beta}{\alpha} = 2.22 \times 10^{-15}\, r. \quad . \; . \; . \; . \; . \; . \; . \quad (96)$$

This ratio is for

Jupiter	0.172
Saturn	0.374
Uranus	0.655
Neptune	1.02
Plato	1.46

The rings near the sun were flat. The smaller the ring the flatter it is. The rings of the smaller planets within the orbit of Jupiter, although the values $\varkappa$ and b have changed, all have been flat. Only those of Uranus, Neptune and Pluto attained a toruslike structure.

This, of course, will aid us to explain some features of these planets with their satellite systems. And here again we meet the 2 flat rings of Saturn, close round their primary, as living proofs of a structure, now clearly understood in its genesis by theory.

It is a stimulating aspect of the account which we have tried to give that the solar system in successive stages of evolution, could be identified with a CARTESIAN whirl, with KANT's disc and the rings of LAPLACE. Even CHAMBERLIN and MOULTON's planetesimals were met at a certain stage, whereas for JEANS' foreign star there might have been work to do at the moment of conception. It might have started the evolution, leaving the rest to be done spontaneously. At any rate, this beautifully ordered structure, which is our planetary system, is essentially self-made.

January 1940.

Anthropology. — *On the Increase of Stature in the Netherlands and the Possibility of its Explanation by Genetic Changes.* By W. A. MIJSBERG (Batavia).

(Communicated at the meeting of March 30, 1940.)

Since the middle of the preceding century the stature of the population of the Netherlands has considerably increased. This fact has been established by studying the measurements taken by the Boards which yearly carry out medical examinations on the male subjects that have reached the age of 19 years. As these examinations have to decide as to the fulfilment of military service they are compulsory; consequently the stature of each male of the said age is taken.

The extensive material on stature brought together in this way has been studied by BRUINSMA [1]), BOLK [2]) and VAN DEN BROEK [3]). As a result of their investigations it can be stated that in the period 1863—1925 the mean stature has continually increased, the total increase during that period being no less than 65 mm (VAN DEN BROEK).

A long time ago already it was established that at the age of 19 about 99 % of the adult stature is attained. Therefore the increase cannot be explained by assuming that nowadays the total stature is reached at an earlier time of life than was the case formerly. One is forced to admit that at present the adults are also taller. According to VAN DEN BROEK in 1921—1925 the mean stature of the conscripts was 170.77 cm. From the figures published by BENDERS (1938) [4]) it appears that since the latter period the increase has not been arrested. There is no reason to suppose that the increase should have been limited to the male part of the population.

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1) G. W. BRUINSMA, Toename in lichaamsbouw der mannelijke bevolking van Nederland. Ned. Tijdschr. v. Geneesk., p. 1495 (1906).

2) L. BOLK, Over de lichaamslengte der mannelijke bevolking van Nederland. Ned. Tijdschr. v. Geneesk., p. 1703 (1909).

L. BOLK, Over de toeneming in lichaamslengte der mannelijke bevolking van Nederland. Ned. Tijdschr. v. Geneesk., p. 650 (1910).

L. BOLK, Ueber die Körperlänge der Niederländer und deren Zunahme in den letzten Dezennien. Zeitschr. f. Morphol. u. Anthrop., **18** (1914).

3) A. J. P. VAN DEN BROEK, Over de voortzetting der toeneming van de lichaamslengte in Nederland. Proc. Kon. Akad. v. Wetensch., Amsterdam, **36** (1930).

A. J. P. VAN DEN BROEK, De anthropologische samenstelling der bevolking van Nederland. Mensch en Maatschappij (1930).

4) A. M. BENDERS, De toeneming der lichaamslengte van de bevolking in Nederland. Mensch en Maatschappij, **14** (1938).

With regard to the explanation of the increase different opinions have been expressed. In discussing them it should be borne in mind that the phenomenon is not confined to the Netherlands; it has been found in all European countries where it has been studied. Consequently the increase cannot be explained by assuming immigration of taller individuals and emigration of shorter ones. Especially for Sweden, the population of which belongs to the tallest of all, such an explanation is impossible. Yet in this country, as is proved by the extensive investigations of HULTKRANTZ (1927) [1]) and LUNDMAN (1939) [2]), an increase of at least 80 mm has occurred in the period 1840—1936. Since in the earlier years of this period the conscripts were measured at the age of 20, the increase must even be higher.

Most authors have expressed the view that improvement of external conditions has caused the increase. It is self-evident that not each individual will reach the maximal adult stature to which his hereditary factors might enable him. For the definite stature is brought about by growth, and all external influences which prevent the hereditary growth-factors from fully displaying their action will cause an adult stature that is shorter than the potential stature of the individual, i.e. the maximal stature that might have been reached on account of his hereditary factors.

Among such external influences insufficient quantity and especially insufficient quality of ingested food, moreover poor health, are of the utmost importance. The improvement of nutrition and of hygienic conditions which is still in progress, as it is spreading in all social layers of the population of the Netherlands, in my opinion [3]) largely accounts for the increasing average stature of the adults. Possibly the stimuli which nowadays in ever increasing number irritate the ears and the eyes and thus stimulate the neuro-glandular apparatus might play a certain part too (IMPERIALI, vide my paper just mentioned).

Accordingly the increase of stature depends on the fact that the mean observed stature of the population is ever more catching up with its mean potential stature.

Not all authors are of this opinion. According to them the mean potential stature has increased too; the latter change might perhaps be the more important one. Among them BENDERS [4]) has given a very definite account

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1) J. V. HULTKRANTZ, Ueber die Zunahme der Körpergrösse in Schweden in den Jahren 1840—1926. Nova Acta Reg. Soc. Scientiarum Upsaliensis. Volumen extra ordinem editum 1927.

2) B. J. LUNDMAN, Ueber die fortgesetzte Zunahme der Körperhöhe in Schweden 1926 bis 1936. Zeitschr. f. Rassenkunde, **9** (1939).

3) W. A. MIJSBERG, Lichaamslengte als anthropologisch kenmerk. Natuurk. Tijdschr. v. Nederlandsch-Indië, **99** (1939).

4) A. M. BENDERS, De toeneming der lichaamslengte van de mannelijke bevolking in Nederland. Ned. Tijdschr. v. Geneesk., p. 1438 (1916).
A. M. BENDERS, De toeneming der lichaamslengte van de bevolking in Nederland. Mensch en Maatschappij, **14** (1938).

of the genetic changes which are said to have taken place. He based his explanation on BOLK's statement that the increase of stature which started in 1863 has been preceded by a decrease from 1821, the earliest year of which data are available, till 1858. If this is true the recent increase cannot be understood without first knowing the cause of the preceding decrease. According to BENDERS the decrease has resulted from the negative selection brought about by NAPOLEON's wars.

Before continuing BENDERS' reasoning I wish to lay stress on the fact that in studying all available data I reached the conclusion that during the period 1821—1858 no decrease of stature has taken place. BOLK's statement is based on part of the material only; if all the data are taken into account it appears, as I pointed out in my above-mentioned paper, that during the whole period in question the average stature of the conscripts was short. Some variability can be observed but no tendency towards greater average stature in the beginning of the said period than at its end can be detected.

Although the starting-point of BENDERS' hypothesis is erroneous I will continue his reasoning since it leads to some genetic questions which I wish to consider in this paper.

According to DAVENPORT [1]) (l.c., p. 315) CHARLES LYELL writing from France in 1828 says the French troops are "...... a stunted race. By accurate calculation of the height of men of the levy since the peace, it is found that the mean height of Frenchmen has been diminished several inches by the Revolution and NAPOLEON's wars. These are now the sons of those who were not thought by NAPOLEON strong and tall enough to fight and look well".

The same negative selection has according to BENDERS taken place in the Netherlands during the French supremacy from 1795 till 1815. In this period only the shorter men were allowed to stay at home, all of them could establish a family in mating with females of normal average potential stature (N-generation); of the taller men part was killed in the battle-fields so that only part of them could marry and establish a family. Consequently the offspring (F_1-generation) of the parents during the said period had an average potential stature intermediate between the averages of the normal mothers and the negatively selected fathers.

By accepting BENDERS' suppositions regarding the age of marriage and the ages of the parents at the times of the births of their eldest and youngest children, it can be demonstrated that the individuals belonging to the F_1-generation appeared for the first time among the conscripts of the year 1820. In the following years the number of F_1-individuals increased till in 1840 all conscripts belonged to the F_1-generation. In this year the lowest average stature of the conscripts should have been observed, for

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[1]) CH. B. DAVENPORT, Inheritance of Stature. Eugenics Record Office. Bulletin No. 18. (Genetics 2) (1917).

starting in 1841 a yearly increasing number of individuals belonging to the N-generation, or after 1844, descending from at least one parent belonging to this generation, was admixed to the F_1conscripts. For further particulars I may refer to my paper "De toeneming van de lichaamslengte der bevolking van Nederland" which will shortly appear in the Journal "Mensch en Maatschappij". In that paper I am setting forth that on the base of BENDERS' hypothesis the mean potential stature of the Dutch conscripts must have been stabilized again about 1870 on a level between that of the N-generation and that of the F_1-generation. Consequently the up till recent times continuing increase of the mean actually observed stature of the conscripts cannot be explained by BENDERS' hypothesis.

BENDERS reached another result. According to him the increase of the average potential stature could not start in 1841; it was postponed till 1866 in consequence of the shortening influence exercised by the admixture of sons of parents both of which belonged to the F_1-generation. These sons, constituting the F_2-generation, according to him should have been extremely short. Now this surmise cannot be correct. For since both the parental generations had the same genetic compound with regard to factors for tallness it seems highly improbable that the average potential stature of their offspring (F_2-generation) should have differed from that of the parental-generations (F_1-generation). Still the question what changes could occur, seems interesting enough to be studied in detail.

For good reasons it may be assumed that the hereditary base of stature is due to the presence in each body-cell of a number of independent, similar and equal factors for tallness, the effect of which is cumulative. If we suppose that in the population no more than 3 pairs of factors can be present there will be seven groups, the members of one group being without factors for tallness, the members of the other groups having 1, 2, 3, 4, 5 and 6 such factors respectively in their body-cells. If we further suppose that in the Dutch population before 1795 the groups presented a normal distribution, the frequencies of the groups will be found by calculating for each group the possible combinations of its number of factors out of the total number of six. As follows from the theory of combinations the frequencies of the 7 groups will correspond with the coefficients of the terms resulting from expanding $(a + b)^6$ (vide table 1).

Since we also want to know in what frequencies different types of germ-cells in which factors up to three in number may be present, are produced by the individuals belonging to each group, it is of advantage to draw up the possible combinations for each group. Let us take as an example the group with 2 factors. Both the factors may be inherited from the father (3 possibilities) or from the mother (again 3 possibilities); but it is also possible that one factor has come from the father, yielding 3 possibilities each of which may occur in combination with the presence of each of the three possible maternal factors (3×3 combinations). The total number of combinations therefore is 15. Only in case each factor has

come from a different parent a pair of factors may be present; this occurs in three cases out of 15. In these cases all germ-cells are provided with one factor. In the other 12 combinations occurring in the body-cells the genes never form pairs; it can easily be computed that in those cases

TABLE 1.

Frequencies of groups with different numbers of factors for tallness in the body-cells in case of normal distribution the highest number of factors being six. Frequencies of germ-cell types in each group and in the population as a whole.

Factors in body-cells to the numbers of:	Frequencies of groups defined in first column	Frequencies of germ-cells with factors to the numbers of:			
		0	1	2	3
0	1 = 1.5625 %	1	—	—	—
I	6 = 9.3750 %	3	3	—	—
II	15 = 23.4375 %	3	9	3	—
III	20 = 31.2500 %	1	9	9	1
IV	15 = 23.4375 %	—	3	9	3
V	6 = 9.3750 %	—	—	3	3
VI	1 = 1.5625 %	—	—	—	1
	64	8	24	24	8

germ-cells without a factor, with one factor and with two factors are produced in the proportions of 1 : 2 : 1 or 3 : 6 : 3. It thus appears that by the group of individuals in whose body-cells two factors are present the said three types of germ-cells (0, 1 and 2) are produced in the proportions of 3 : 9 : 3. In table 1 these proportions have been set forth and the same has been done with regard to the other groups. By adding up the numbers it appears that in the whole population the four types of germ-cells are produced in the proportions of 1 : 3 : 3 : 1.

Now I do not wish to pretend that the Dutch population before 1795 should necessarily have possessed an entirely normal distribution of the seven groups in question. Probably some deviations from normal conditions were present, but, as will be understood presently, such deviations, if not too considerable, do not interfere with the general trend of our calculations.

Now according to BENDERS, during the period of the French supremacy a certain number of the taller men was killed. Consequently among the men who could form an offspring the groups with a low number of factors for tallness in their body-cells were more frequent than before. Let us suppose that all the men belonging to the groups 0 and I were too short and were therefore exempted from military service, whereas of each of the groups II to VI one third of the men was killed; then the proportions of the different groups among the men who could produce offspring were as represented in the second column of table 2.

The frequencies with which the different types of germ-cells occur in

each group can be calculated in the same way as in table 1. The resulting proportions of germ-cell types in all male parents appear from table 2.

TABLE 2.

Frequencies of groups with different numbers of factors for tallness in their body-cells among males having married in the period 1795—1815. Frequencies of germ-cell types in each group and in all these males together.

Factors in body-cells to the numbers of:	Frequencies of groups defined in first column	Frequencies of male germ-cells with factors to the numbers of:			
		0	1	2	3
0	3	3	—	—	—
I	18	9	9	—	—
II	30	6	18	6	—
III	40	2	18	18	2
IV	30	—	6	18	6
V	12	—	—	6	6
VI	2	—	—	—	2
	135	20	51	48	16

These males married women in which the distribution of germ-cell types was normal. Male germ-cells without factors for tallness occurring in a frequency 20 out of 135 can fertilize female germ-cells of the four types in the proportions of 1 : 3 : 3 : 1. In order to avoid fractions the probabilities of such fertilizations are calculated for $8 \times 20 = 160$ male germ-cells of the 0-type out of a total number of 1080. The results are put down in table 3 (first vertical series of products), from which table the frequencies with which the other combinations occur may be seen too. By adding up the products horizontally, the frequencies with which in the F_1-generation the seven groups with different numbers of factors for tallness in their body-cells occur, can be computed (last column of table 3).

In comparison with normal distribution (vide table 1) the frequencies of the groups 0, I and II are higher in the F_1-generation, those of the other groups lower. The cause of these deviations appears from the first factors of the partial products in table 3.

It will be clear at once that the average potential stature of the F_1-generation is lower than that of the N-generation. In the latter it is equal to the potential stature of the individuals of group III. If it is supposed that to every additional factor present is due an equal increase of the potential stature by k units, the average potential stature of the F_1-generation is that of the individuals of group III minus $.055556\,k$. In table 4 this has been expressed as follows: III — $.055556\,k$.

If now one wishes to calculate the frequencies of the different groups in the F_2-generation (offspring of F_1-parents) it is necessary to compute first the proportions in which the four germ-cell types are produced in the

F_1-generation. In order to do so it should be borne in mind that the individuals of the F_1-generation belonging to group III, taking these as an instance, have come into existence in four different ways with four different frequencies, as is illustrated by the partial products occurring in the corresponding horizontal row of table 3. In each case the proportions

TABLE 3.
F_1-generation. Frequencies of groups with different numbers of factors for tallness in their body-cells. These figures hold good for both sexes.

Factors in body-cells to the numbers of:	Frequencies of unions of the different types of parental germ-cells	Frequencies of groups defined in first column
0	20×1	$20 = 1.851852\,\%$
I	$20 \times 3 + 51 \times 1$	$111 = 10.277778\,\%$
II	$20 \times 3 + 51 \times 3 + 48 \times 1$	$261 = 24.166667\,\%$
III	$20 \times 1 + 51 \times 3 + 48 \times 3 + 16 \times 1$	$333 = 30.833333\,\%$
IV	$51 \times 1 + 48 \times 3 + 16 \times 3$	$243 = 22.500000\,\%$
V	$48 \times 1 + 16 \times 3$	$96 = 8.888889\,\%$
VI	16×1	$16 = 1.481481\,\%$
		1080

in which the four germ-cell types are produced have to be calculated separately. After doing this for all the partial products in table 3, the adding of the figures found gives the frequencies of the germ-cell types produced by the F_1-generation as a whole. It goes without saying that in both sexes the distribution is the same. The frequencies of the body-cell groups in the F_2-generation can now easily be found. Table 3 shows the way in which it is done. Since the distribution of germ-cell types is the same in both parental generations, some of the partial products are equal two by two, whereas of others the two factors are the same.

In order to find the distribution of body-cell groups in the F_3-generation (offspring of F_2-parents) and following generations the operations described should be repeated. Their most complicated part consists in establishing the distribution of germ-cell types in the next generation. Therefore it is of advantage to give this part of the calculations in a more generalized form as follows. If in both parental generations the 4 types of germ-cells are produced in the proportions of $p : q : r : s$, then in the next filial generation the same germ-cell types are produced in the following proportions:

$$
\begin{aligned}
&0)\quad 12\,p^2 + 12\,pq + 6\,pr + 3\,ps + 2\,q^2 + qr.\\
&1)\quad 12\,pq + 12\,pr + 9\,ps + 8\,q^2 + 11\,qr + 6\,qs + 2\,r^2.\\
&2)\quad 6\,pr + 9\,ps + 2\,q^2 + 11\,qr + 12\,qs + 8\,r^2 + 12\,rs.\\
&3)\quad 3\,ps + qr + 6\,qs + 2\,r^2 + 12\,rs + 12\,s^2.
\end{aligned}
$$

The sum of all terms is $12\,(p + q + r + s)^2$.

This formula corresponds with the one given by PHILIPTSCHENKO [1]) (p. 271). I derived the formula before knowing his, which testifies of the correctness of the two. Moreover in taking $p:q:r:s = 1:3:3:1$ the proportions in the F-generation are the same as in the parental generations, as should be the case in ideal distribution.

The formula may also be used in case the distribution of germ-cell types in one parental generation is in the proportions of $p:q:r:s$ and in the other of $a:b:c:d$; in such a case p^2 must be substituted by pa, pq by $(pb + qa):2$ etc.

By means of this formula I calculated the procentual frequencies of the seven groups which can be distinguished after the number of factors for tallness in the body-cells, in the F_2-, F_3- and F_4-generations. In table 4 these data have been given together with the corresponding data already known regarding the N- and F_1-generations.

TABLE 4.

Factors in body-cells to the numbers of :	Procentual frequencies of groups defined in first column				
	N	F_1	F_2	F_3	F_4
0	1.562500	1.851852	1.799448	1.771268	1.757653
I	9.375000	10.277778	10.181866	10.134001	10.109829
II	23.437500	24.166667	24.230955	24.265451	24.281572
III	31.250000	30.833333	31.024729	31.120814	31.169216
IV	23.437500	22.500000	22.529084	22.544258	22.552802
V	9.375000	8.888889	8.793362	8.745180	8.720954
VI	1.562500	1.481481	1.440556	1.419028	1.407974
Mean potential Stature	III	III — .055556 k	III — .055556 k	III — .055556 k	III — .055556 k

It appears from table 4 that in the generations which come after the F_1-generation, the frequencies of the extreme groups (0, I, V and VI) decrease, whereas those of the more central groups show an increase. The changes are greatest in passing from F_1 to F_2; in passing to younger generations they become smaller at every turn. *Notwithstanding these changes the mean potential stature remains the same in all generations succeeding the F_1-generation.*

The correctness of the latter important conclusion can be more directly proved in the following way. It was supposed that in the paternal and in the maternal generations the four types of germ-cells are produced in the proportions of $p:q:r:s$; consequently the average number of factors for tallness present in the germ-cells is $(q + 2r + 3s) : (p + q + r + s)$.

[1]) J. PHILIPTSCHENKO, Ueber Spaltungsprozesse innerhalb einer Population bei Panmixie. Zschr. f. indukt. Abstamm. und Vererb.lehre, **35** (1924).

Since fertilization occurs according to chance the average number of factors present in the body-cells of the members of the F_1-generation will be twice as high. In the production of germ-cells the number of factors is divided into halves; therefore the mean number of factors present in the germ-cells of the F_1-generation must be equal to that of the parental generations. Thus it is evident that no change of the mean potential stature can occur in following generations.

In order to check the results obtained I repeated the calculations, starting from the assumption that the maximal number of factors for tallness to be found in the body-cells may be 5 pairs. In case of normal distribution the frequencies of the 11 groups comprising individuals with 0—XI factors in the body-cells respectively are given by the coefficients of the terms resulting from developing $(a + b)^{10}$, i.e.: 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1. If it is supposed that during the French supremacy all male individuals of the groups 0—III could establish a family and produce offspring, whereas only half the male representants of the other groups could do so, the said eleven groups will have occurred in the male parental generations of that period in the proportions of $2 : 20 : 90 : 240 : 210 : 252 : 210 : 120 : 45 : 10 : 1$. In the same way as before the frequencies of the six male germ-cell types can be calculated; in the female generations the latter types are produced in the proportions of $1 : 5 : 10 : 10 : 5 : 1$ (normal distribution).

Again it is useful to derive a formula giving the frequencies of the six germ-cell types in the F_1-generation in case these frequencies in both parental generations are in the proportions of $p : q : r : s : t : u$. In this case the six types of germ-cells are produced in the F_1-generation in the following proportions:

0) $160\,p^2 + 160\,pq + 80\,pr + 40\,ps + 20\,pt + 10\,pu + 32\,q^2 + {}$
$+ 24\,qr + 8\,qs + 2\,qt + 3\,r^2 + rs.$

1) $160\,pq + 160\,pr + 120\,ps + 80\,pt + 50\,pu + 96\,q^2 + 136\,qr + 80\,qs + {}$
$+ 42\,qt + 20\,qu + 36\,r^2 + 29\,rs + 8\,rt + 3\,s^2.$

2) $80\,pr + 120\,ps + 120\,pt + 100\,pu + 32\,q^2 + 136\,qr + 144\,qs + 116\,qt + {}$
$+ 80\,qu + 82\,r^2 + 130\,rs + 80\,rt + 40\,ru + 36\,s^2 + 24\,st.$

3) $40\,ps + 80\,pt + 100\,pu + 24\,qr + 80\,qs + 116\,qt + 120\,qu + 36\,r^2 + {}$
$+ 130\,rs + 144\,rt + 120\,ru + 82\,s^2 + 136\,st + 80\,su + 32\,t^2.$

4) $20\,pt + 50\,pu + 8\,qs + 42\,qt + 80\,qu + 3\,r^2 + 29\,rs + 80\,rt + {}$
$+ 120\,ru + 36\,s^2 + 136\,st + 160\,su + 96\,t^2 + 160\,tu.$

5) $10\,pu + 2\,qt + 20\,qu + rs + 8\,rt + 40\,ru + 3\,s^2 + 24\,st + 80\,su + {}$
$+ 32\,t^2 + 160\,tu + 160\,u^2.$

The sum of these terms is $160\,(p + q + r + s + t + u)^2$.

With the help of the formula the frequencies of the 11 groups characterized by the number of factors for tallness in their body-cells have been calculated in case of normal distribution (N), in the sons and daughters of N-mothers and *War*-fathers (F_1), in the F_2-generation (offspring of F_1-parents) etc. The results have been laid down in table 5. From its

TABLE 5.

Factors in body-cells to the numbers of:	Procentual frequencies of groups defined in first column				
	N	F_1	F_2	F_3	F_4
0	.097656	.151042	.148020	.143361	.140647
I	.976562^{5}	1.380208	1.344186	1.315598	1.300278
II	4.394531	5.625000	5.511122	5.449975	5.419483
III	11.718750	13.567708	13.458859	13.425304	13.411081
IV	20.507812^{5}	21.614583	21.712964	21.784201	21.821805
V	24.609375	23.932292	24.206068	24.334278	24.396863
VI	20.507812^{5}	18.750000	18.899491	18.955219	18.980493
VII	11.718750	10.286458	10.209534	10.168141	10.146921
VIII	4.394531	3.776042	3.652546	3.595257	3.567434
IX	.976562^{5}	.833333	.781352	.756679	.744857
X	.097656	.083333	.075858	.071987	.070138
Mean potential Stature	V	V — .175000 k	V — .175000 k	V — .175000 k	V — .175000 k

figures it appears that a decrease of the frequencies of the extreme groups (0, I, II, III, VII, VIII, IX and X) occurs in the generations succeeding the F_1-generation; in passing to younger generations the changes gradually grow less considerable. The mean potential stature however remains fixed at the same distance below the potential stature of the individuals bearing 5 factors in their body-cells. The difference is .175000 k, the latter constant being the increase of potential stature due to the presence of each additional factor in the body-cells.

From his calculations PHILIPTSCHENKO has drawn the conclusion that as a result of the mixture of two different populations or of two layers of the same population which formerly did not mix, in the following generations increase or decrease of the numbers of individuals which in certain respects are far below or above the average may be expected to occur. This conclusion, which is of great importance in questions regarding race-crossing, has been affirmed by my own calculations. In my examples a decrease of the frequencies of the extreme groups occurred, but PHILIPTSCHENKO has demonstrated that in case the initial frequencies differ in other directions from normal distribution other changes may be seen. He even deduced formulas from which the changes to occur in following generations may be predicted.

In the course of his conclusions PHILIPTSCHENKO also stated (l.c., p. 277) that the decrease or the increase of the mean stature of the population which has been found in some countries, may be explained in the same way. This is the only reference to the problem of the increase of stature as well as to changes occurring in mean values to be found in his paper. Therefore one would be inclined to pass it over in silence if it had not found its way into the literature on the subject. Thus I am obliged to lay stress on the fact that PHILIPTSCHENKO did not give any proof whatever of the correctness of this statement, whereas it appears from my own calculations that although the frequencies of the different groups change in following generations, the mean potential stature remains fixed at the value it has in the F_1-generation.

It is HULTKRANTZ that felt inclined to attach much value to PHILIPTSCHENKO's explanation of the increase of stature in consequence of genetic changes brought about by mixture and BENDERS (1938) also mentioned it on HULTKRANTZ' authority. Both these authors lay stress on the mathematical base of PHILIPTSCHENKO's deductions, but unfortunately the latter author omitted the mathematic probing of only this one statement.

Another genetic explanation of the increase of the average stature of a population might possibly consist in the occurrence of heterosis resulting from mixture of different parts or layers of the population which previously did not mix. According to HULTKRANTZ (l.c., p. 50) WAALER, in a study on the Norwegian population, mentioned this possibility. HULTKRANTZ thinks its foundation insufficient, but according to LUNDMAN (l.c., p. 269) it is on the whole accepted by the Swedish geneticists.

By heterosis (= hybrid vigor = "Luxurieren der Bastarde") the well-known fact is expressed that crosses between different strains or races of plants or animals may produce offspring more vigorous in many respects than either parent type. It is most markedly in evidence when two highly inbred strains, i.e. strains consisting mainly of homozygous individuals, are crossed.

The explanation seems to be that the character in question (in our case stature) is the result of a number of cumulative factors present in both parental generations with the exception of at least one pair which is dissimilar in the two parental generations or is present in one of them only. Now all the individuals of the F_1-generation will be heterozygous with regard to the factors occurring in homozygous conditions in one of the parents only. If it is further assumed that the presence of one factor of the pair has the same effect on the phenotype as the pair of them has, it is evident that the main stature of the F_1-generation will exceed either of the parental generations in case each of them possesses at least one pair of factors not present in the other.

There are, however, many objections to explaining the increase of stature of a human population in this way.

1. Stature is a character which in the population presents a high variability similar to that of normal distribution. The genetic base of such variability is the presence of multiple factors which have a cumulative effect even when present as a pair. The assumption of genes which in heterozygous and in homozygous conditions have the same effect on the phenotype does not fit in with this kind of variability.

2. In case of heterosis in the F_2-generation (offspring of heterozygous F_1-parents) and following ones, the homozygous conditions will be partially restored which will cause a decrease of the mean stature; this result is well-known in plant breeding. Of course for the time being this decrease could be overcompensated by increase due to heterosis resulting from new hybridizations. Still very short individuals would continue to occur in younger generations as well as they did before. Now according to BOLK (l.c.) the minimal stature of 2000 non-Jewish conscripts in the town of Amsterdam was 120 cm in 1850, whereas in 1900 it was 144 cm. Among the Jewish population of this town (750 individuals examined) the lower limit of the range of variability rose during the same period from 126 cm to 144 cm!

3. If heterosis should occur in man it ought to be most clearly seen in crossings between individuals of two widely different races of mankind. It was FISCHER [1]) who, in studying the Rehobother Bastards, reached the conclusion that in some respects, especially in stature, the average of the hybrids surpassed those of both the European paternal and the Hottentot maternal ancestors. He ascribed this result to heterosis and expressed the view that BOAS' finds in North-American hybrids descending from European fathers and Indian mothers were due to the same cause. The great trouble with such investigations is that nothing definite is known of the average stature of the male ancestors. The investigator is therefore compelled to assume that their mean stature corresponded with that of the population from which they had emigrated. But it is of course quite possible that the emigrants belonged to the taller part of the population. The same holds good with regard to the female ancestors.

More recently FISCHER appears to be not so sure of the occurrence of heterosis in man as he was before. In 1930 he wrote [2]) (p. 184): "Man darf keinesfalls aus der Feststellung eines Luxurierens in zwei Fällen (BOAS und FISCHER — einmal angenommen, dass es wirklich ein Luxurieren ist) den Schluss ziehen, es müsse jede Rassenkreuzung zu Luxurieren führen". The latter warning refers to the circumstance that in other cases of race-crossing no signs of heterosis could be detected. But even in case a bastardpopulation should be taller than expected, FISCHER gives the

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[1]) E. FISCHER, Die Rehobother Bastards und das Bastardierungsproblem beim Menschen. Jena (1913).

[2]) E. FISCHER, Versuch einer Genanalyse des Menschen, mit besonderer Berücksichtigung der anthropologischen Systemrassen. Zschr. f. indukt. Abst. u. Vererb.lehre, **54** (1930).

following warning (l.c., p. 183): "Dabei möchte ich aber heute viel mehr als seiner Zeit bei meinen Bastarduntersuchungen betonen, dass man mit der Annahme echten Luxurierens sehr viel zurückhaltender sein muss. Einmal erklären uns die Annahme von Allelen und von Siebungsvorgängen (SCHEIDT) manche Fälle, die zuerst ein Luxurieren andeuten könnten...... etc.".

All in all the occurrence of heterosis in man seems far from being proved; moreover as explained before, with regard to the problem of the increase of stature, it does not seem a very successful hypothesis.

In the foregoing quotation FISCHER refers to two publications by SCHEIDT. In the first [1]), p. 134, SCHEIDT describes as "Luxurieren der Mischlinge" the fact that in a family some of the children may exceed either parent in some respects or fall behind them. Such a fact can be easily explained by multiple factors ("Allelen"). Only these individual variations do not explain the problem we are interested in, viz. the excess shown by the bastard population as a whole, as apparent from its average stature. It seems that FISCHER overlooked this difference.

In the second paper [2]) SCHEIDT emphasizes the onset of puberty as the event which puts an end towards growth. If in a population puberty sets in at a rather early time of life the population will stop growing early, before its members have reached their entire potential stature; consequently the average adult stature will be rather short. Here SCHEIDT is in accordance with DAVENPORT's earlier statement.

BOLK [3]) however found 1923 that in Dutch women the onset of puberty in the younger generation of his time occurred about $1\frac{1}{2}$ years earlier than it did in the preceding generation. According to FISCHER (l.c., p. 196) similar anticipation was found by STEIN in Germany (Freiburg) and by SCHREINER in Norway.

Still it may be safely accepted that the younger generation of Dutch women is on an average not shorter than the preceding one; on the contrary the continuous increase of stature proved in males seems to have occurred in the females as well. Consequently the onset of puberty must not be regarded as the important event which controls the extent to which the potential stature can be realized. It seems more probable that in the younger generation a more rapid growth occurred which permitted puberty to set in at an earlier time of life.

Thus SCHEIDT's efforts to explain the skewness of the frequency-

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[1]) W. SCHEIDT, Allgemeine Rassenkunde. München (1925).

[2]) W. SCHEIDT, Die Asymmetrie der Körpergröszenkurven und die Annahme der Poymerie. Arch. f. Rass.- u. Gesellsch.biol., **16** (1925).

[3]) L. BOLK, De Menarche bij de Nederlandsche vrouw en de vervroeging ervan bij de jongste generatie. Proc. Kon. Akad. v. Wetensch., Amsterdam, **32**, No. 7 (1923).

L. BOLK, Statistisch onderzoek over de Menarche bij de Nederlandsche bevolking. Geneesk. Bladen, **24**, No. 6 (1925).

curves of stature present in many populations and the increase of the average stature of the same by assuming recessive factors which stop growth before the individuals have reached their ultimate potential stature, are fruitless.

DAVENPORT's statement according to which persons of similar stature tend to marry each other interferes with the results of my calculations thus far, that the relative frequencies of the different groups of potential stature in following generations may differ from the figures given by me in the tables 4 and 5, since I supposed matings to occur according to chance only (panmixia). But it does not interfere with the principal result of my calculations, viz. the constancy of the mean potential stature in following generations. The more direct proof I gave of it remains valid. I emphasize this fact since SCHEIDT, by combining his hypothesis of recessive genes putting an untimely end to growth with DAVENPORT's selective matings, tried to prove the opposite.

I may conclude by saying that up till now the increase of the average stature of populations has not been satisfactorily explained from a genetic point of view. In my opinion such an explanation could only be given by proving that short individuals on an average produce less offspring than tall ones do. But I do not know of investigations indicating that bachelors and single women on the whole should be shorter than heads of families and married women, or that short parents on an average should get fewer children than tall parents.

Physics. — *Structure and* ZEEMAN-*effect of doubly ionized Thorium, Th. III.* By T. L. DE BRUIN and P. F. A. KLINKENBERG. (Communicated by Prof. P. ZEEMAN.)

(Communicated at the meeting of April 27, 1940.)

Introduction.

The work concerning the structure of the spectra of Thorium was begun in the laboratory "Physica" in 1935. At that time nothing was known about the spectral structure and even the separation between the lines belonging to the different stages of ionization was never investigated. Since we have made a new description of the spectra of thorium. New measurements of wavelengths have been made. An investigation concerning the absorption spectra with the under water spark method and the method of explosion of wires have been completed [1]). Farther extensive data of ZEEMAN-effects have been obtained [2]).

In Th. III a scheme of levels was detected giving the j and g values. However it was not possible to identify the multiplets because the lines of Th. III mainly are in the ultra violet and the ZEEMAN-effects obtained with the grating mounting, are not sufficiently resolved. LANG has found independently many important $\triangle \nu$ differences however without indicating j values [3]).

By means of new data obtained with a LUMMER plate it was possible to identify the terms originating from the most important electron configurations of the doubly ionized atom.

Spectrum of Thorium III.

The thorium III atom is an example of a two electron spectrum and terms of the configurations fd, fs, fp, ds, pd, d^2 etc. can be expected. It was however uncertain if the spectrum would have a structure similar to La. II [4]) and Ce. III [5]) or to Ti. III [6]) and Zr. III [7]). This was one of the

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[1]) T. L. DE BRUIN and J. N. LIER, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **41**, 956 (1938).

[2]) J. N. LIER, Thoriumspectra en hun ZEEMAN-effect. Thesis, Amsterdam, June 1939.

[3]) R. J. LANG, Phys. Rev. **56**, 272. August 1939.

[4]) RUSSELL and MEGGERS, Nat. Bur. Stand. J. Research, **9**, 664 (1932).

[5]) T. L. DE BRUIN, J. N. LIER and H. J. V. D. VLIET, Proc. Kon. Akad. v. Wetensch.. Amsterdam, **40**, 334 (1937).

H. N. RUSSELL, R. B. KING and R. J. LANG, Phys. Rev. **52**, 456 (1937).

H. J. V. D. VLIET, Het ZEEMAN-effect van de spectraallijnen van Cerium en Neodymium. Thesis, Amsterdam, Januari 1939.

[6]) RUSSELL and LANG, Astr. J. **66**, 13 (1927).

[7]) C. C. KIESS and R. J. LANG, Nat. Bur. Stand. J. Res. **5**, 305 (1930).

questions to clear up. The analysis of the spectrum will also give data to check the theory of complex spectra developed by CONDON and SHORTLEY [8]) based on the quantum mechanics. Farther the analysis of the Th. III spectrum can give a start for the analysis of the iso-electronic sequence Ra. I, Ac. II, Th. III, Pa. IV, U. V and the knowledge of electron coupling for the elements at the end of the periodic system.

The following terms can be expected:

Configuration	Abrev.	Terms	Number	Parity
5 f 7 s	fs	$^3F_{234}$ $^1F_3$	4	odd
5 f 6 d	fd	$^3H_{456}$ $^3G_{345}$ $^3F_{234}$ $^3D_{123}$ $^3P_{012}$ $^1H_5$ $^1G_4$ $^1F_3$ $^1D_2$ $^1P_1$	20	odd
6 d 7 p	dp	$^3F_{234}$ $^3D_{123}$ $^3P_{012}$ $^1F_3$ $^1D_2$ $^1P_1$	12	odd
5 f 7 p	fp	$^3G_{345}$ $^3F_{234}$ $^3D_{123}$ $^1G_4$ $^1F_3$ $^1D_2$	12	even
6 d 7 s	ds	$^3D_{123}$ $^1D_2$	4	even
6 d^2	d^2	$^3F_{234}$ $^3P_{012}$ $^1G_4$ $^1D_2$ $^1S_0$	9	even

As might be expected for so heavy an atom the separations of the components of the singlets and triplet terms are very wide. The actual values of j have been determined from the ZEEMAN-effects.

Experimental Procedure.

The *lightsource* we used was the ordinary BACK vacuum trembler. As experience taught us the lines of the thorium spectrum were narrow enough with this source to be analysed by interferometric methods. Certainly this is due to the high mass of the Th.-atom giving a small DOPPLER breadth and to the absence of any hyperfine structure or isotope effect.

The employed *magnet* was the great WEISS-magnet of the laboratory. It was used in a circuit permitting us to vary the current from 1 to 100 Ampère. The constance of the current was carefully controlled by means of a variable shunt resistance.

The *spectroscopic apparatus* consisted of a LUMMER plate of crystal quartz (thickness 4.75 mm, length 150 mm) and of a HILGER E 1 spectrograph. The LUMMER plate was placed in a metal box the interior of which is covered with sheets of cork in order to procure a suitable thermal insulation. Moreover the room temperature was kept constant within 0°.2 C. by means of an automatic temperature control. Under this circum-

8) CONDON and SHORTLEY, Phys. Rev. **37**, 1025 (1931).

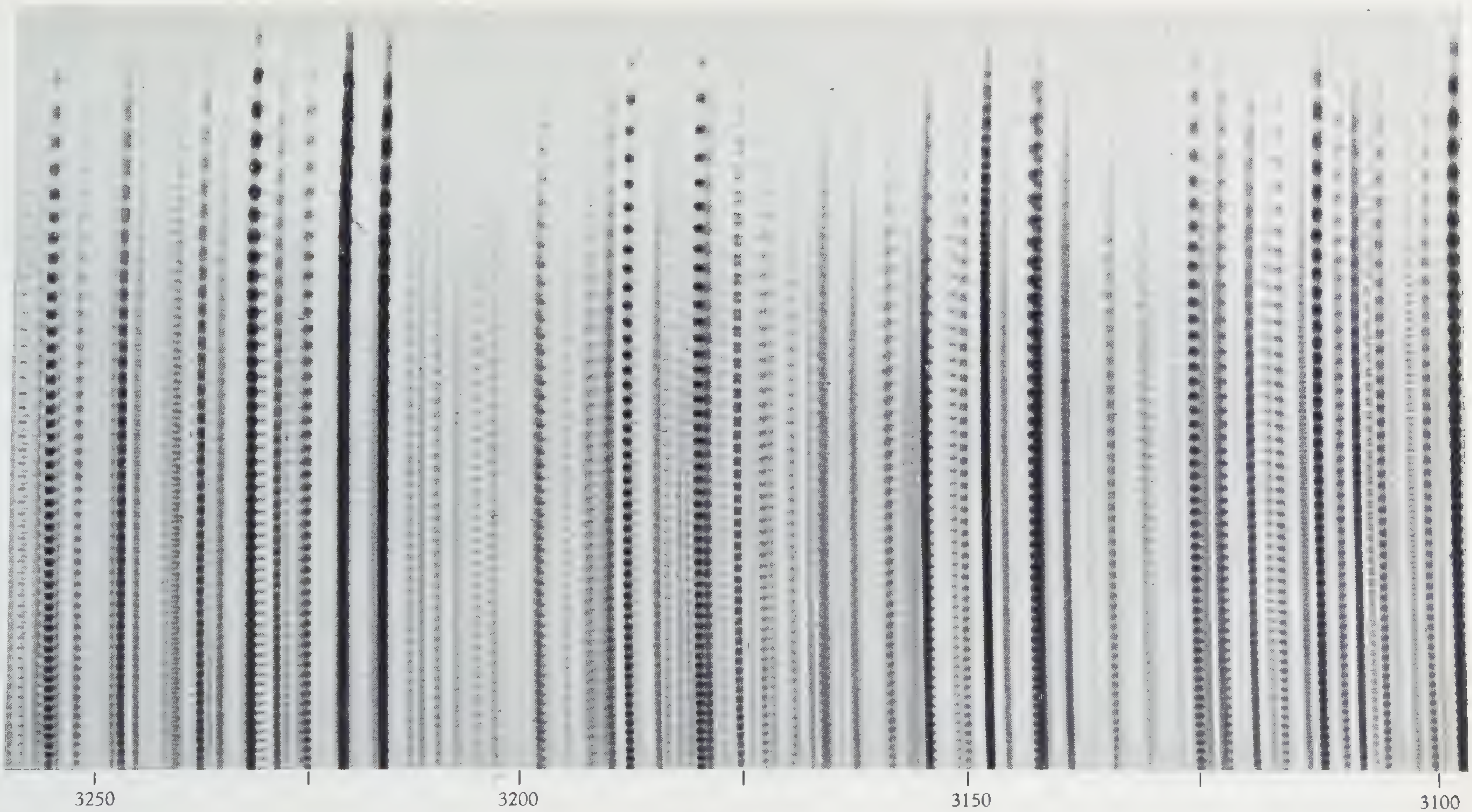

Fig. 1.
Part of the thorium spectrum at 22870 gauss.

stances it was possible to limit the temperature variations of the LUMMER plate to 0°.02 C. which was sufficiently small for our purpose.

The light coming from the source was separated into the two polarisations by a calcspar rhomb and then thrown on the prism of the LUMMER plate. The interferences were formed by a 250 mm quartz-fluorite achromatic lens and focussed on the slit of the spectrograph.

The separation of the two polarisations was necessary for two reasons:

1°. The double refraction of the LUMMER plate giving rise to a little displacement of the maxima belonging to one polarisation with respect to that belonging to the other [9]).

2°. The particularly small spectral range of the LUMMER plate, not more than 5—8 times the real breadth of the maxima belonging to the Th. III-lines.

Fig. 1 shows an enlarged part of a photograph; it represents the π-polarisation of the Th. spectrum between 3270 and 3095 Å obtained in a magnetic field of 22870 gauss.

From this picture it will be clear that even in the separated polarisations it is impossible to determine the real ZEEMAN-patterns from one plate alone, for the manyfold coincidences of different orders disturb the intensities completely and even the number of components can be considerably higher than is shown on the photograph. With more complicated ZEEMAN-splittings it is easy to be seen that the field being high enough the whole spectral range will be filled up continuously and no structure will be observed. The real ZEEMAN-pattern can only be derived from the behaviour of the interference pattern in a gradually increasing field. A more detailed description of the method will be given elsewhere.

A series of photographs was made at fields of 5970, 11960, 13108, 20720, 22870, 24320, 29820 and 37760 gauss in the region from 3470 to 2460 Å. and another series at fields of 9190, 26150, 38580 and 39120 gauss in the region from 7000 to 3220 Å. The times of exposure were in general 1 to 2 hours on Ilford Special Rapid Plates; the stronger lines were already fully exposed in about 10 minutes. The current strength of the lightsource was 2—3 Ampère. Pure Th.-metal was used as the anode.

At carefully overlooking the photographs taken at different field strengths the observation was made that with increasing field the lines belonging to higher states of ionisation gradually become more pronounced such that on the plates taken at 29000—38000 gauss the Th. I and Th. II-lines were very weak compared with the spectrograms at fields of 0—12000 gauss whereas the Th. III-lines remain very strong. The Th. IV line λ 2693.97 is still considerably more intensified than the neighbouring Th. III lines. This change of the intensity distribution can be seen in fig. 2*a* and *b* representing the same wavelength interval of the Th. spectrum (3221—2973 Angström) at 5970 and at 37760 gauss respectively.

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[9]) J. H. GISOLF, Proc. Kon. Akad. v. Wetensch., Amsterdam, **38**, 735 (1935).

The effect is so clear that it could be employed as a method to determine the state of ionisation; in this way we recognised the Th. III lines λ 4048.87, 3610.40, 3516.82, 3394.55, 3284.35, 3122.40, 3083.24, 3033.13, 2999.23, 2938.19, 2917.75, 2889.00, 2811.38, 2785.73, 2776.84, 2761.55 and 2667.95. Many of them were placed in the Th. II-list before and even one in the Th. I list. For most of these lines the observation could be checked by examining the spectrum of the vacuum arc and that of air sparc in the last of which the Th. III lines are intensified also; moreover some of them were found to be combinations of newly detected terms of Th. III.

The line 3046.95, on the contrary, given before as Th. III, belongs to Th. II or Th. I as it is much weakened in the field.

Measurement of the Photographs.

The distances in the interference pattern were measured generally in 8—14 orders depending from the wavelength; then the real frequency differences were evaluated by means of the method described by MacNair [10]), expressing the separation in terms of the spectral range. This was computed from the formula

$$\Delta\nu = \frac{\sqrt{\mu^2 - 1}}{2d\left(1 - \mu^2 + \lambda\mu\dfrac{\partial\mu}{\partial\lambda}\right)} \quad {}^{11)}$$

For the refraction index μ that of the extra-ordinary ray ε was put when measuring the π-polarisation and that of the ordinary ray ω when measuring the σ-polarisation.

At first the field strengths were measured with the cadmium lines photographed on each plate before and after every Th.-exposure. But it turned out that the Cd. lines did not give consistent values probably in consequence of hyperfine structure and isotope shift. The separations appeared to be not proportional to the field strenghts. As was observed later on the Zn. triplet λ 4810, 4722, 4680 is much more suitable as these lines are sharper and lie in a region where the distance of the interference orders is fairly high. After this a few exposures of Th. and Zn. were made at fields of 38730 and 25820 gauss and then a number of Th. lines throughout the whole region studied were measured against Zn. The Th. lines were carefully chosen and only quite sharp triplets and quartets were used in order to avoid complications from coincidences. These lines were finally employed to determine the field strengths and of the so obtained data for each plate the average value was taken. At 37760 gauss the values were consistent within 0.2 % and at 5970 gauss within 1.0 %.

Owing to the very high dispersion of the Lummer plate the broadening of the unresolved π-polarisation in lower fields could be measured with

[10]) W. A. MacNair, Phil. Mag. **2**, 613 (1926).
[11]) O. v. Baeyer, Phys. Z. S. **9**, 831 (1908).

T. L. DE BRUIN and P. F. A. KLINKENBERG: STRUCTURE AND ZEEMAN-EFFECT OF DOUBLY IONIZED THORIUM, TH. III.

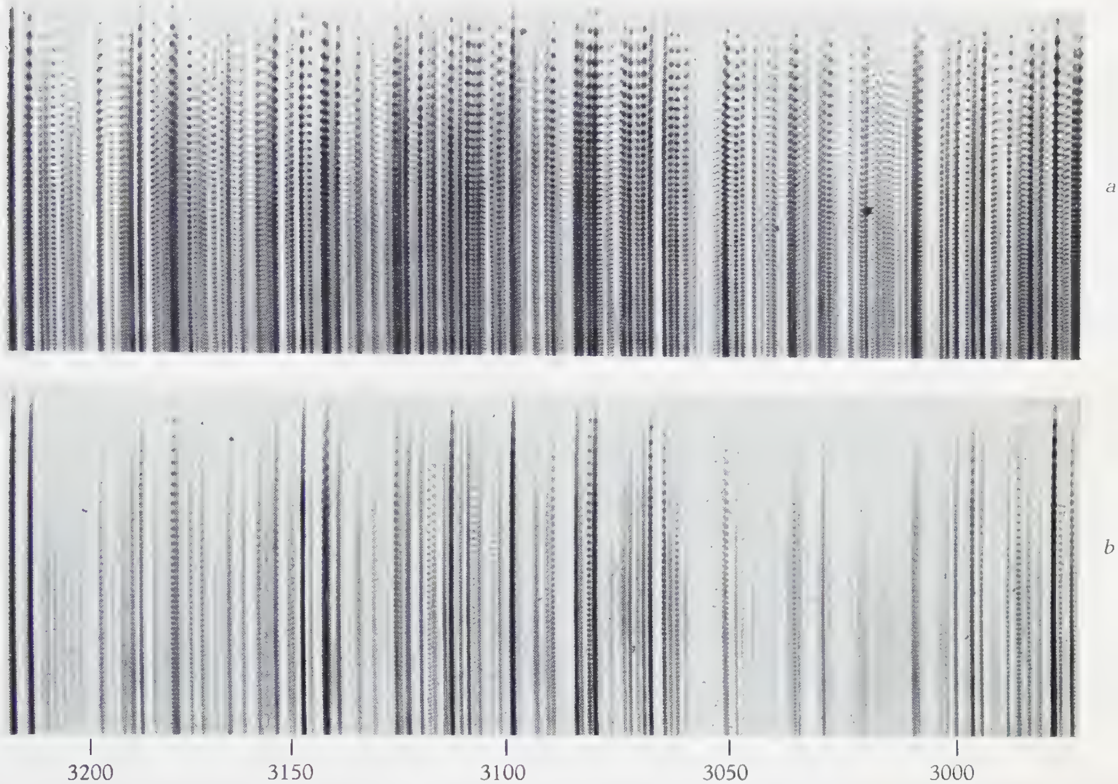

Fig. 2.
Thorium lines at 5970 and 37760 gauss.

Fig. 3.

sufficient accuracy to decide upon the number of components if their mutual distances were known. These were taken from the plates showing a sharply resolved pattern. Often this is the case for more than one photograph showing the line at different stages of overlapping. The obtained ZEEMAN-pattern can then be controlled by the characteristic change of the intensity curve.

Fig. 3*a*, *b* and *c* show the π-polarisation of the line λ 3028.57 of Th. II. At 6000 gauss (*a*) we see a quartet, at 12000 gauss (*b*) an unsharp doublet and at 29800 gauss (*c*) we see no structure at all; the separation of the π-components has become as wide as the spectral range and all components coincide. This is a case where the opening of the π-polarisation is very wide so that we see the real structure already at 6000 gauss.

In consequence of the lack of resolving power of the LUMMER plate for the light of which the electric vibration component lies in the vertical plane the measurement of the σ-components generally is less accurate than that of the π-components. So it often occurs that the differences of the g-values can be determined accurately whereas their absolute values are less certain. Not entering into details here we can only state that by taking into account all the peculiarities of the interference pattern at different field strengths

TABLE I. Relative Terms in the Th III spectrum.

Electr. conf.	Term	Level	g	Electr. conf.	Term	Level	g
fd	$^3H_4$	0.00	0.87	fp	$^3D_2$	43757.98	0.99
fs	$^3F_2$	511.00	0.74	fp	$^3D_1$	44601.70	0.51
fd	$^3H_5$	2496.35	1.00	fp	$^3G_4$	47260.10	1.14
fs	$^3F_3$	2526.43	1.06	fp	$^3G_5$	47420.60	1.21
fd	$^3F_2$	3180.91	0.72	fp	$^3D_3$	47470.28	1.21
fd	$^3F_3$	4826.48	1.00				
fd	$^3G_3$	5060.21	0.87	dd	$^3F_2$	52385.86	0.74
fd	$^1D_2$	6288.00	0.91	dd	$^3F_3$	56379.50	1.08
fs	$^3F_4$	6309.58	1.23	ds	$^1D_2$	57000.00	1.01
fs	$^1F_3$	7499.85	1.03	ds	$^3D_1$	57846.75	0.49
fd	$^3P_0$	7866.78	0/0	dd	$^3F_4$	58861.73	1.20
fd	$^3D_1$	7920.78	0.62	ds	$^3D_2$	59499.68	1.17
fd	$^3F_4$	8141.25	1.11	ds	$^3D_3$	62277.00	1.34
fd	$^3H_6$	8436.35	1.17	dp	$^3F_2$	89603.76	0.78
fd	$^3G_4$	8980.20	1.19	dp	$^3D_1$	91603.68	0.90
fd	$^3D_3$	10740.97	1.25	dp	$^3D_2$	96412.07	1.20
fd	$^3P_1$	11122.60	1.37	dp	$^1F_3$	96514.20	1.12
fd	$^3G_5$	11276.62	1.20	dp	$^3D_3$	96788.70	1.12
fp	$^1D_2$	28233.00	1.13	dp	$^1P_1$	97387.17	1.11
fp	$^3G_3$	33561.63	0.84	dp	$^1D_2$	100002.79	1.03
fp	$^3F_2$	34995.75	0.82	dp	$^3P_2$	101166.95	1.53
fp	$^1F_3$	38430.62	1.18	dp	$^3F_3$	102303.42	1.19
fp	$^1G_4$	38579.86	1.14	dp	$^3P_1$	103315.79	1.22
fp	$^3F_3$	42312.44	0.97	dp	$^3F_4$	105374.56	1.26
fp	$^3F_4$	43700.76	1.07				

TABLE 2. Th III lines.

Int.	λ	ν	Classification	Observed ZEEMAN-effect
6	4555.64	21944.68	fd $^1D_2$ — fp $^1D_2$	(0.22) (**0.46**) 0.65 **0.89** **1.14** 1.38 [1]
2	3990.57	25052.01	fd $^3F_2$ — fp $^1D_2$	(0.39) (**0.81**) 0.34 **0.71** **1.14** 1.53 [1]
5	3692.40	27074.97	fd $^3D_1$ — fp $^3F_2$	(**0**) (0.21) 0.62 0.82 **1.03** [1]
5	3661.48	27303.61	fd $^3G_5$ — fp $^1G_4$	(0) ...... **1.50** E = 0.32; G = 1.33
3	3610.40	27689.89	fd $^3D_3$ — fp $^1F_3$	(0.21) 1.22 cy [1]
4	3606.17	27722.38	fs $^3F_2$ — fp $^1D_2$	(0.36) (**0.76**) 0.38 **0.75** **1.12** 1.50 [1]
5	3591.05	27839.09	fd $^3D_3$ — fp $^1G_4$	(0) 0.97 ay [1]
15	3507.57	28501.64	fd $^3G_3$ — fp $^3G_3$	(0.063) 0.85_7
3	3394.55	29950.56	fd $^3G_4$ — fp $^1F_3$	(0) 1.21_5 E = 0.08
10	3377.42	29599.93	fd $^3G_4$ — fp $^1G_4$	(0.285) 1.13
6	3339.56	29935.48	fd $^3G_3$ — fp $^3F_2$	(0) 0.899 cy; E = 0.14
4	3320.85	30104.14	ds $^1D_2$ — dp $^3F_2$	(0.36) (**0.72**) – **0.86** **1.22** –
16	3313.68	30169.28	fd $^3F_3$ — fp $^3F_2$	(**0**) (0.18_9) (0.38) – – – 1.18 **1.370**
14	3300.50	30289.74	fd $^3F_4$ — fp $^1F_3$	(0) 1.00_5 ay; E = 0.21
18	3290.59	30380.97	fd $^3F_2$ — fp $^3G_3$	(**0**) (0.12_5) (0.25) 0.56 0.68_5 0.81 0.93_5 **1.06**
16	3232.06	30931.12	fs $^1F_3$ — fp $^1F_3$	(0.14) (0.28) (**0.427**) 1.10_5
20	3221.22	31035.20	fs $^3F_3$ — fp $^3G_3$	(0.22) (0.44) (**0.655**) 0.40 0.62 **0.841** **1.059** 1.28 1.50
16	3216.57	31080.07	fs $^1F_3$ — fp $^1G_4$	(0) 1.25_6 cy; E = 0.30
16	3148.00	31757.03	ds $^3D_1$ — dp $^1F_2$	(**0**) (0.29_1) – 0.79 **1.07**
4	3142.28	31814.84	fd $^3F_2$ — fp $^3F_2$	(0.176) 0.78
4	3114.00	32103.75	ds $^3D_2$ — dp $^3D_1$	(**0**) (0.280) 0.90 1.18 **1.45_5**
14	3112.33	32120.98	fs $^3F_4$ — fp $^1F_3$	(0) 1.30_6 y; E = 0.20
8	3097.95	32270.07	fs $^3F_4$ — fp $^1G_4$	...... (**0.43**) 1.24 [2]
6	3083.24	32424.02	fd $^3G_5$ — fp $^3F_4$	(0) ...... **1.64** cy; E = 0.50; G = 1.41
12	3078.90	32469.73	fs $^3F_3$ — fp $^3F_2$	(**0**) (0.24) (0.49) – – – 1.27 **1.52_8**
12	3066.25	32603.68	ds $^1D_2$ — dp $^3F_2$	(0.23) (**0.46**) 0.55 **0.78** **1.01** 1.23
4	3063.27	32635.40	fd $^3P_1$ — fp $^3D_2$	
3	3033.13	32959.68	fd $^3D_3$ — fp $^3F_4$	(0) 0.76 y; E = 0.46
4	3027.85	33017.15	fd $^3D_3$ — fp $^3D_2$	(**0**) (0.25_2) (0.50) – – 1.25 1.50 **1.75_0**
4	2999.23	33332.20	fd $^3G_4$ — fp $^3F_3$	(0) 1.09_3 cy; E = 0.14
8	2995.78	33370.59	fd $^3G_3$ — fp $^1F_3$	(–) (0.57) (**0.88_0**) – [3]
6	2986.07	33479.09	fd $^3P_1$ — fp $^3D_1$	(0.87_4) 0.48 1.36
20	2978.75	33561.36	fd $^3H_4$ — fp $^1D_2$	(0) 0.941 y; E = 0.14
8	2974.94	33604.34	fd $^3F_3$ — fp $^1F_3$	(–) (–) (**0.46**) – [4]
4	2961.81	33753.31	fd $^3F_3$ — fp $^1G_4$	(0) 1.39 y; E = 0.30
6	2961.51	33756.92	ds $^3D_1$ — dp $^3D_1$	(0.40_7) 0.48_3 0.90_0
8	2928.67	34135.23	ds $^3D_3$ — dp $^3D_2$	(**0**) (0.16) (0.32) 0.92 1.08 1.24 1.40 **1.56**
6	2925.60	34171.05	fd $^3F_4$ — fp $^3F_3$	(**0**) (0.15) (0.29) (0.44) – – – – – 1.33 **1.48** [5]
18	2898.97	34484.93	fs $^3F_2$ — fp $^3F_2$	(0.12_5) 0.77
14	2896.70	34511.95	ds $^3D_3$ — dp $^3D_3$	(0.21) (0.43) (**0.643**) 0.70 0.92 **1.130** **1.344** 1.56 1.77
3	2889.00	34603.93	ds $^1D_2$ — dp $^3D_1$	(0) 1.08_7 E = 0.11
4	2871.70	34812.38	fs $^1F_3$ — fp $^3F_3$	(0.142) 0.999
7	2811.38	35559.27	fd $^3F_4$ — fp $^3F_4$	(0.113) 1.084
8	2784.37	35904.19	fs $^3F_3$ — fp $^1F_3$	(0.28_6) 1.12
8	2778.23	35983.54	fd $^3G_5$ — fp $^3G_4$	(0) 1.35 y; E = 0.27
7	2776.84	36001.55	fs $^3F_4$ — fp $^3F_3$	(**0**) (0.26) (0.51) (0.76) – – – – – 1.70 **1.96**
8	2775.05	36024.77	fd $^1D_2$ — fp $^3F_3$	(0) 1.037 E = 0.12
7	2765.93	36143.55	fd $^3G_5$ — fp $^3G_5$	(0) 1.198 b; E = 0.09
3	2761.55	36200.48	fs $^1F_3$ — fp $^3F_4$	(0) 1.134 E = 0.07

TABLE 2. *(Continued.)*

Int.	λ	ν	Classification	Observed ZEEMAN-effect
3	2757.21	36257.85	fs $^3F_3$ — fp $^3D_2$	(0) 1.07 y; E = 0.09
3	2725.40	36681.02	fd $^3D_1$ — fp $^3D_1$	(0.130) 0.562
3	2721.82	36729.40	fd $^3D_3$ — fp $^3D_3$	(0) 1.25 z; E = 0.11
4	2721.40	36734.92	fd $^3P_0$ — fp $^3D_1$	(0) 0.51 E = 0.04
8	2708.39	36911.38	ds $^3D_2$ — dp $^3D_2$	(0) 1.190 b; E = 0.10
6	2700.86	37014.28	ds $^3D_2$ — dp $^1F_3$	(0) 1.083 y; E = 0.17
15	2686.07	37218.08	d^2 $^3F_2$ — dp $^3F_2$	(0) 0.771 E = 0.11
10	2680.95	37289.15	ds $^3D_2$ — dp $^3D_3$	(0) 1.075 E = 0.11
7	2673.65	37390.95	fs $^3F_4$ — fp $^3F_4$	(–) (–) (0.41) (**0.55**) 1.15
4	2667.95	37470.84	fd $^1D_2$ — fp $^3D_2$	(0.132) 0.96
6	2666.88	37485.87	fd $^3F_3$ — fp $^3F_3$	(0.080) 0.98_7
7	2638.60	37887.60	ds $^3D_2$ — dp $^1P_1$	(0) 1.211 E = 0.07
6	2635.87	37926.85	d^2 $^3F_4$ — dp $^3D_3$	(0) 1.325 cy; E = 0.21
7	2609.21	38314.35	fd $^1D_2$ — fp $^3D_1$	(**0**) (0.403) 0.51 0.92 **1.320**
20	2600.62	38440.62	fd $^3G_4$ — fp $^3G_5$	(0) 1.244 E = 0.07
20	2597.31	38489.73	fd $^3G_4$ — fp $^3D_3$	(0) 1.159 E = 0.07
3	2587.20	38640.27	fd $^3G_3$ — fp $^3F_4$	
10	2583.35	38697.85	fd $^3G_3$ — fp $^3D_2$	(0) **0.64** .... ay; $E = 0.33$; $G = 0.73$
25	2571.61	38874.50	fd $^3F_3$ — fp $^3F_4$	(0) 1.22 y; E = 0.23
10	2567.82	38931.88	fd $^3F_3$ — fp $^3D_2$	(0) 1.018 E = 0.10
25	2564.37	38984.25	fd $^3H_6$ — fp $^3G_5$	(0) 1.067 y; E = 0.12
15	2554.73	39131.35	fd $^3F_2$ — fp $^3F_3$	(**0**) (0.25) (0.49) – – – 1.25 **1.50**
8	2549.12	39217.46	d^2 $^3F_2$ — dp $^3D_1$	(0) 0.651 y; E = 0.21
15	2545.09	39279.55	fd $^3F_4$ — fp $^3G_5$	(0) 1.513 y. E = 0.45
5	2541.86	39329.46	fd $^3F_4$ — fp $^3D_3$	(0) 0.88_9 y; E = 0.34
12	2536.54	39411.89	ds $^1D_2$ — dp $^3D_2$	(0.19) (**0.37**$_2$) 0.81 **1.00** **1.21** 1.40
12	2529.96	39514.44	ds $^1D_2$ — dp $^1F_3$	(0) 1.238 y; E = 0.20
4	2528.31	39540.23	ds $^3D_1$ — dp $^1P_1$	(0.64_3) 0.82
20	2514.31	39760.38	fs $^1F_3$ — fp $^3G_4$	(0) .... **1.50** cy; E = 0.40; G = 1.40
25	2512.69	39786.01	fs $^3F_3$ — fp $^3F_3$	(0.256) 1.000
15	2512.52	39788.78	ds $^1D_2$ — dp $^3D_3$	
30	2501.08	39970.68	fs $^1F_3$ — fp $^3D_3$	(0.18) (0.36) (**0.540**) – 0.84 **1.02** **1.20** 1.38 –
12	2497.60	40026.37	ds $^3D_3$ — dp $^3F_3$	(0.36) 1.25
12	2497.22	40032.46	d^2 $^3F_3$ — dp $^3D_2$	(0) 0.96 y; E = 0.26
15	2475.29	40387.07	ds $^1D_2$ — dp $^1P_1$	(0) 0.95_1 E = 0.05
20	2473.94	40409.12	dd $^3F_3$ — dp $^3D_3$	(0.106) 1.103
30	2463.70	40577.08	fd $^3F_2$ — fp $^3D_2$	(0.509) 0.95
20	2441.27	40949.87	fs $^3F_4$ — fp $^3G_4$	(0.39) 1.18 [6])
15	2431.70	41111.02	fs $^3F_4$ — fp $^3G_5$	(0) 1.21
10	2428.77	41160.60	fs $^3F_4$ — fp $^3D_3$	(0) 1.24
40	2427.96	41174.33	fs $^3F_3$ — fp $^3F_4$	(0) 1.10
10	2424.59	41231.55	fs $^3F_3$ — fp $^3D_2$	(0) 1.06
30	2413.50	41421.00	fd $^3F_2$ — fp $^3D_1$	(0) 0.94
15	2399.25	41666.99	ds $^3D_2$ — dp $^3P_2$	(0.65) 1.35
30	2391.53	41801.49	fs $^3F_2$ — fp $^3F_3$	(0) 1.15
15	2371.41	42156.12	ds $^3D_1$ — dp $^1D_2$	
10	2368.96	42199.71	fd $^3G_3$ — fp $^3G_4$	(0) 1.44
10	2357.22	42409.86	fd $^3G_3$ — fp $^3D_3$	(0.92) 1.06
10	2355.90	42433.62	fd $^3F_3$ — fp $^3G_4$	(0) 1.24

TABLE 2. *(Continued.)*

Int.	λ	ν	Classification	Observed ZEEMAN-effect
4	2344.28	42643.94	fd $^3F_3$ — fp $^3D_3$	(0) 0.11
20	2335.52	42803.87	ds $^3D_2$ — dp $^3F_3$	(0) 1.22
15	2324.72	43002.70	ds $^1D_2$ — dp $^1D_2$	(0) 1.03
20	2319.62	43097.24	ds $^3D_3$ — dp $^3F_4$	(0) 1.16
10	2311.58	43247.13	fs $^3F_2$ — fp $^3D_2$	(0.47) 0.81
20	2301.22	43441.81	dd $^3F_4$ — dp $^3F_3$	(0) 1.20
20	2291.65	43623.20	dd $^3F_4$ — dp $^1D_2$	(0) 1.16
10	2281.57	43815.91	ds $^3D_2$ — fp $^3P_1$	(0) 1.20
5	2270.66	44026.42	dd $^3F_2$ — dp $^3D_2$	
4	2263.42	44167.23	ds $^1D_2$ — dp $^3P_2$	
15	2221.47	45001.19	dd $^3F_2$ — dp $^1P_1$	(0) 0.95
25	2206.65	45303.39	ds $^1D_2$ — dp $^3F_3$	(0) 1.25
25	2198.60	45469.24	ds $^3D_1$ — dp $^3P_1$	(0.78) 0.85
15	2176.86	45923.92	dd $^3F_3$ — dp $^3F_3$	
30	2149.25	46513.16	dd $^3F_4$ — dp $^3F_4$	
15	2099.42	47616.34	dd $^3F_2$ — dp $^1D_2$	

Notes:

[1]) Taken from J. N. LIER, thesis.
[2]) Centre of gravity of $\pi = 0.36$; σ only approximative.
[3]) Asymmetric splitting; σ-components not measurable.
[4]) σ-components not measurable; according to LIER the separation should be 1.13.
[5]) Position of the σ-components approximative.
[6]) Below 2450 Angström the given separations are taken from LIER (measurements on quartz spectrograph).

List of Abbreviations used in table 2:

a = σ-components show decrease of intensity to the outside.
b = σ-components show decrease of intensity on both sides.
c = σ-components show decrease of intensity to the inner side.
y = π-component shows decrease of intensity to the outside.
z = π-component shows decrease of intensity to the inner side.
E = the *edge* of the unresolved π-component.
G = the centre of gravity of the unresolved σ-component.

we generally succeeded in effecting a very satisfactory accuracy even with complicated ZEEMAN-patterns. The simpler patterns could nearly always be determined with a high degree of exactness.

Termtable and list of classified lines.

The identified terms of the Th. III spectrum are collected in table 1. Higher series members are to be expected to give lines still farther in the ultra violet. Therefore it was not possible to determine the absolute term-values. Four groups of energy levels have been detected. The lowest group

of levels in analogy with the structure of Ce. III has been interpreted as belonging to the odd $5f7s$ and the $4f6d$ electron configuration. The middle levels come from the even $5f7p, 6dd, 6d7s$ configuration. The high levels are due to the $6d\,7p$ electron configuration. Many of the combinations between the *dd, ds* and *fd, fs* terms can be expected in the farther ultra violet. It should be noticed that the terms have only an scale based on estimations. The observed *g*-values of the terms are also listed in table 1. The *fs* and *ds* configurations have practically normal *g*-values. The *fd, fp* and *dp* configurations however show anomalous coupling. Doubt can be arise about the identification of the $fs\,{}^3F_{23}$ and $fd\,{}^3F_{23}$. It is possible that the symbols should be interchanged. It would be possible to settle this question by detecting the deep ${}^2F$ level in Th. IV [12]). In the *fd* configuration some terms are still lacking. In analogy with Ce. III it can be expected that the ${}^1G_4$ term is still deeper than the detected energy levels.

Comparison with other spectra.

In the first place a comparison can be made with the two electron systems La. II and Ce. III. The relative arrangement of the levels belonging to the same configuration is strikingly similar in La. II, Ce. III and Th. III. However the binding of the analogous configurations is not the same. For instance the *fs* configuration in Th. III compared with the *fd* terms is much deeper than in Ce. III. As an example we compare in table 3 the *g*-values of analogous terms in La. II, Ce. III and Th. III. Further it should be noticed that in the iso electronic sequence Ra. I, Ac. II, Th. III in Ra. I the lowest configurations are s^2, *sp* and *sd*. In Th. III however so far as known *fd* and *fs*.

TABLE 3.

	LS	La II	Ce III	Th III		LS	La II	Ce III	Th III
fs ${}^3F_2$	0.67	0.67	0.66	0.74	fd ${}^1G_4$	1.00	1.00	0.99	....
fs ${}^3F_3$	1.08	1.09	1.07	1.06	fd ${}^3F_2$	0.67	0.76	0.76	0.72
fs ${}^3F_4$	1.25	1.26	1.27	1.23	fd ${}^3H_4$	0.80	0.86	0.87	0.87
fs ${}^1F_3$	1.00	1.05	1.03	1.03	fd ${}^3F_3$	1.08	1.09	1.10	1.00
fp ${}^3G_3$	0.75	0.87	0.87	0.84	fd ${}^3G_3$	0.75	0.77	0.76	0.87
fp ${}^3F_2$	0.67	0.73	0.82	0.82	fd ${}^3H_5$	1.03	1.08	1.07	1.00
fp ${}^1F_3$	1.00	0.95	1.16	1.18	fd ${}^1D_2$	1.00	0.93	0.88	0.91
fp ${}^3F_3$	1.08	1.06	0.95	0.97	fd ${}^3F_4$	1.25	1.24	1.30	1.11
fp ${}^3F_4$	1.25	1.14	1.09	1.07	fd ${}^3G_4$	1.05	1.06	1.06	1.19
fp ${}^3D_2$	1.17	1.10	0.99	0.99	fd ${}^3H_6$	1.17	1.17	1.17	1.17
fp ${}^3D_1$	0.50	0.49	0.40	0.51	fd ${}^3D_1$	0.50	0.55	0.60	0.62
fp ${}^3G_4$	1.05	1.15	1.13	1.14	fd ${}^3G_5$	1.20	1.20	1.19	1.20
fp ${}^3D_3$	1.33	1.31	1.24	1.21	fd ${}^3D_2$	1.17	1.19	1.19	....
fp ${}^3G_5$	1.20	1.20	1.21	1.21	fd ${}^3D_3$	1.33	1.32	1.29	1.25
fp ${}^1G_4$	1.00	1.06	1.05	1.14	fd ${}^3P_0$	0/0	0/0	0/0	0/0
fp ${}^1D_2$	1.00	1.05	1.08	1.13	fd ${}^3P_1$	1.50	1.46	1.29	1.37

12) R. J. LANG, Can. J. Res. 14, 43 (1936).

Summary.

With the interferometric method of quartz LUMMER plate crossed with a quartz spectrograph the ZEEMAN-effect of the thorium lines has been investigated. A number of 50 energy levels of the doubly ionized atom Th. III have been detected. A list of classificd Th. III lines is given. The structure of the Th. III spectrum is not analogous to Ra. I but to Ce. III. The g-values have been compared.

Laboratory "Physica" of the University of Amsterdam.

April, 1940.

Mathematics. — *Self-projective point-sets.* By Dr. O. BOTTEMA. (Communicated by Prof. W. VAN DER WOUDE).

(Communicated at the meeting of April 27, 1940.)

1. If we consider in n-dimensional space S_n a set of $(n+2)$ points (no $n+1$ of which belong to a S_{n-1}) taken in a given order, there always exists a non-singular collineation, which interchanges the points of the set in a given way. This is a consequence of the well-known fact that a collineation is determined by giving $(n+2)$ pairs of conjugated points. The theorem does not hold for a set of $(n+3)$ points (no $n+1$ of which belong to a S_{n-1}) taken in a given order and which may be called a *throw* (dutch: *worp*). If we exclude the case $n=1$ (the four points then being permutable according to the *Vierergruppe*) there does not generally exist a collineation, differing from identity, so that the set, taken as a whole, is not altered. The question arises to construct-analogous with harmonic and equi-anharmonic sets in the case $n=1$-throws which are invariant for certain finite collineation groups, thus showing „projective symmetry" which the general throw lacks.

For $n=2$ and $n=3$ the question was completely solved by BARRAU [1]). Answering a prize-question of the Wiskundig Genootschap for the year 1938 following BARRAU's line of thought I gave additional remarks to the general theory.

In the following by a new method a complete solution is given.

2. As invariants for the throw $A_1, A_2, \ldots A_{n+3}$, BARRAU takes the set of homogeneous coordinates $(a_1; a_2; \ldots a_{n+1})$ of the point A_{n+3} with regard to a system where the first $(n+1)$ points are fundamental points and A_{n+2} is the unit-point. The classes of projective throws are thus represented by the points of an S_n. By the $(n+3)!$ permutations of the $(n+3)$ points of the throw, the invariants a_i are transformed in a well-defined way and take, for $n>1$, in general $(n+3)!$ values. In the S_n an involution of degree $(n+3!$ is created. The coinciding points of the involution represent the self-projective throws.

The method here given is based on the well-known fact, that the $(n+3)$ points of a throw in S_n always lie on a rational normal curve C_n of degree n; the curve is uniquely determined by the points. If t is the rational parameter on C_n, the points of the throw can be given by

[1]) BARRAU, Proc. Kon. Akad. v. Wetensch., Amsterdam **39**, 955—961 (1936); **40**, 150—155 (1937).

a set of $(n+3)$ values of t. We suppose that $t=t_i$ corresponds with the point A_i. If $(pqrs)$ stands for the anharmonic ratio

$$\frac{(s-q)(r-p)}{(r-q)(s-p)},$$

we consider the set of n non-homogeneous values

$$p_i=(t_{n+3}\,t_{n+2}\,t_{n+1}\,t_i). \qquad (i=1,2,\ldots n)$$

According to their geometrical meaning the p_i are invariants for the projective group in S_n. They are a complete set of invariants for the throw: two throws which have the same set of p_i are projective. This follows immediately from the following theorems:

1°. All the C_n in S_n are projective.

2°. A C_n is invariant for a group of ∞^3 collineations in S_n, which corresponds with the group of linear transformations of the rational parameter.

It can easily be shown, that the p_i notwithstanding their different origin are not essentially unlike BARRAU's invariants. If $A_1, A_2, \ldots A_{n+1}$ are taken as fundamental points for the system of coordinates, a C_n which passes through these points has the equations

$$x_i=B_i\,\frac{P(t)}{(t-t_i)} \qquad (i=1,2,\ldots n+1),$$

where B_i are constants and $P(t)$ stands for

$$(t-t_1)(t-t_2)\;\ldots\;(t-t_{n+1}).$$

If the curve moreover passes through the points A_{n+2} and A_{n+3}, the corresponding parameter-values being t_{n+2} and t_{n+3}, the coordinates of these points are respectively

$$x_i'=B_i\,\frac{P(t_{n+2})}{(t_{n+2}-t_i)} \qquad (i=1,2,\ldots n+1)$$

and

$$x_i''=B_i\,\frac{P(t_{n+3})}{(t_{n+3}-t_i)} \qquad (i=1,2,\ldots n+1)$$

If we take A_{n+2} as unit-point the homogeneous coordinates of A_{n+3}, being the BARRAU-invariants a_i of the set, are obviously

$$a_i=\frac{x_i''}{x_i'}=\frac{P(t_{n+3})}{P(t_{n+2})}\cdot\frac{t_{n+2}-t_i}{t_{n+3}-t_i},$$

or

$$a_i=\varrho\,\frac{t_i-t_{n+2}}{t_i-t_{n+3}}$$

where ϱ is a constant.

We have therefore

$$p_i = \frac{a_i}{a_{n+1}},$$

the new invariants thus being shown to be the ratios of BARRAU's.

3. If we have a self-projective throw in S_n, there exists a group of collineations which interchanges the separate points, leaving the throw as a whole invariant. The collineations then leave invariant also the rational normal curve C_n which passes through the $(n+3)$ points and therefore belong to the group of ∞^3 collineations having that property. The latter group is isomorphic with the group of linear transformations in one variable. Therefore the finite group of collineations which leaves the throw invariant is isomorphic with a group of linear transformations for the rational parameter t and can be produced by the latter.

Thus in order to find a self-projective throw, we must consider a set of parameter-values t_i $(i=1, 2, \ldots n+3)$, which has the property that there exists a group of linear transformations of t by which the set as a whole is not altered. Now our investigation is highly facilitated by the fact, that the theory of linear groups in one variable was developed a long time ago. As KLEIN has pointed out, there are no other groups but the cyclic groups (of order k), the dihedron groups (of order $2k$), the tetrahedron group (of order 12), the octahedron group (of order 24) and the icosahedron group (of order 60). In consequence of this we shall not proceed by summing up the possible self-projective throws for a given value of n, but we start from a given group and construct the point-sets which are invariant for this group. For this construction we can make use of the idea of *Diskontinuitätsbereich* as defined by KLEIN. The points of the complex t-plane by means of a stereographic projection can be represented by the points of a sphere the rotations of which correspond with the linear transformations of t. Finite groups of these transformations correspond with groups of rotations belonging to regular polyhedra inscribed in the sphere. The symmetry-planes of these polyhedra divide the surface of the sphere in a number of regions. A point of the sphere, submitted to the rotations of a group of order r, generates a set of points the number of which is r (respectively a factor of r) if the original point belongs to one region (respectively to two or more regions).

4. First considering the cyclic groups and choosing $t=0$ and $t=\infty$ as fixed points, the group is given by

$$t' = \varepsilon_k^{\nu}\, t \qquad (\nu = 0, 1, \ldots k-1),$$

where ε_k stands for $e^{\frac{2\pi i}{k}}$. From an arbitrary point (not coinciding with a fixed point) arises a set of k points, which is invariant for the cyclic group. Thus a point-set which remains unaltered by the group consists

of mk, $mk+1$ or $mk+2$ points, where $m \geqslant 0$ stands for an integer. As t-values of the points of the set we can take $a_\mu \varepsilon_k^\nu$ ($\mu = 1, 2, \ldots m$; $\nu = 0, 1, \ldots k-1$) to which are added 0, 1 or 2 of the points $t=0$ and $t=\infty$. The numbers a_μ can be chosen arbitrarily with the exception of the points of the set having to be different.

It is possible that the sets thus obtained are invariant for a wider group of transformations than the cyclic group. If $k=2$, $m=1$ or $m=2$, we have sets of four points, which are invariant for the Vierergruppe mentioned above; if k is arbitrary, $m=1$, the sets containing k or $k+2$ points are invariant for a dihedron group of order $2k$.

5. The dihedron group can be generated by adding to the cyclic group $t' = \varepsilon_k^\nu t$ the transformation $t' = \frac{1}{t}$. It contains besides the cyclic group the transformations $t' = \frac{\varepsilon_k^\nu}{t}$, whose second powers are the unity-transformation. They interchange the points $t=0$ and $t=\infty$. If an arbitrary point is submitted to the transformations of the group, it generates a set of $2k$ points, which is invariant for the group. This number is reduced to k if the original point is chosen in one of the points $t=\varepsilon_k^\nu$, and to 2 if the point is chosen in $t=0$ or $t=\infty$.

Thus a point-set which is invariant for a dihedron group of order $2k$ consists of

$$m \,.\, 2k + m_1 \,.\, k + m_2 \,.\, 2$$

points; here m is an integer $\geqslant 0$, m_1 is 0 or 1, m_2 is 0 or 1.

It is possible that the point-sets so obtained are invariant for a wider group than the dihedron one. So for $k=4$, $m=0$, $m_1=1$, $m_2=1$, we have a set of six points ($t=1$, i, -1, $-i$, 0, ∞) which is the stereographic projection of an octahedron and accordingly is not altered by the octahedron group.

6. The t-values of a point-set which is the stereographic projection of a tetrahedron, an octahedron or an icosahedron and the formulae for the linear transformation groups which belong to them, are not given here. They can o.g. be found in KLEIN's classical monography [1]).

As for the tetrahedron group, it is of order 12. An arbitrary point, submitted to the group, generates a set of 12 points, the t-values of which can be written down by means of KLEIN's formulae. The number decreases to 6, if the original point is taken on the boundary of two Diskontinuitätsbereiche; the six points correspond to the middles of the edges of the tetrahedron and have the octahedron symmetry. The

[1]) KLEIN, Vorlesungen über das Ikosaeder. Leipzig (1884).

number decreases to 4, if the original point is taken on the boundaries of three regions, the points being the vertices of a tetrahedron (which can be chosen in exactly two ways); they correspond with four values of t which have an equianharmonic ratio.

A point-set which is invariant for the tetrahedron group thus consists of

$$12\,m + 6\,m_1 + 4\,m_2$$

points, where m is an integer $\geqslant 0$, $m_1 = 0$ or 1, $m_2 = 0$, 1 or 2. In the case $m = m_2 = 0$, $m_1 = 1$, the set has the symmetry of the octahedron group.

The octahedron group is of order 24. An arbitrary point induces a set of 24 points, invariant for the group. Choosing the original point in a particular way, there arise sets of respectively 12, 8 and 6 points, each having one and but one representative.

A point-set which is invariant for the octahedron group therefore consists of

$$24\,m + 12\,m_1 + 8\,m_2 + 6\,m_3$$

points, where m is an integer $\geqslant 0$, m_1 is 0 or 1, m_2 is 0 or 1, m_3 is 0 or 1.

The icosahedron group has the order 60. An arbitrary point generates an invariant set of 60 points. There are three particular sets, which contain respectively 30, 20 and 12 points. Accordingly a point-set, invariant for the group consists of

$$60\,m + 30\,m_1 + 20\,m_2 + 12\,m_3$$

points; m is an integer $\geqslant 0$, m_1, m_2 and m_3 are each seperately 0 or 1.

7. Making use of these results, we are able to construct in S_n, where n has an arbitrary vaule, all the sets (containing a finite number of points $r \geqslant n + 3$), which are invariant for a group of projective transformations. For the present we leave the value of n out of account and investigate the sets of r values of t, which are as a whole invariant for the linear groups of t-transformations mentioned above. This being done and t_ν ($\nu = 1, 2, \ldots r$) being such a set, we distribute these values over a rational normal curve C_n in S_n, e.g. the curve with the parameter-equations

$$x_\mu = t^\mu \qquad (\mu = 0, 1, \ldots n).$$

Thus we obtain the throws in S_n, which are self-projective. In what follows we confine ourselves to $r = n + 3$, this case being the starting-point of our investigation.

8. If $n=2$, we must have throws of 5 points. Considering the linear groups, we have the following possibilities:

Cyclic groups.

$m=1$, $k=4$; $t_1=1$, $t_2=i$, $t_3=-i$, $t_4=-1$, $t_5=0$. The points of the throw are: $A_1\equiv(1,1,1,1)$, $A_2\equiv(1,i,-1,-i)$, $A_3\equiv(1,-i,-1,i)$, $A_4\equiv(1,-1,1,-1)$, $A_5\equiv(1,0,0,0)$. The throw is invariant for a cyclic group of 4 collineations, which permutates the points according to the cycle $(A_1\,A_2\,A_3\,A_4)$. $m=1$, $k=2$; $t_1=a_1$, $t_2=-a_1$, $t_3=a_2$, $t_4=-a_2$, $t_5=0$ $(a_1\neq 0,\ a_2\neq 0,\ a_1^2\neq a_2^2)$; $A_1\equiv(1,a_1,a_1^2,a_1^3)$, $A_2\equiv(1,-a_1,a_1^2,-a_1^3)$, $A_3\equiv(1,a_2,a_2^2,a_2^3)$, $A_4\equiv(1,-a_2,a_2^2,-a_2^3)$, $A_5\equiv(1,0,0,0)$. The throw is invariant for a group of order 2, the points being permutated according $(A_1\,A_2)\,(A_3\,A_4)\,A_5$.

Dihedron groups.

$k=3$, $m=0$, $m_1=1$, $m_2=1$; $t_1=\varepsilon_3$, $t_2=\varepsilon_3^2$, $t_3=1$, $t_4=0$, $t_5=\infty$; $A_1\equiv(1,\varepsilon,\varepsilon^2,1)$, $A_2\equiv(1,\varepsilon^2,\varepsilon,1)$, $A_3\equiv(1,1,1,1)$, $A_4\equiv(1,0,0,0)$. $A_5\equiv(0,0,0,1)$ $(\varepsilon=\varepsilon_3)$. The throw is invariant for a dihedron group of order 6, the generating permutations being $(A_1\,A_2\,A_3)\,A_4\,A_5$ and $A_1\,(A_2\,A_3)\,(A_4\,A_5)$. $k=5$, $m=0$, $m_1=1$, $m_2=0$; $t_1=\varepsilon_5$, $t_2=\varepsilon_5^2$, $t_3=\varepsilon_5^3$, $t_4=\varepsilon_5^4$, $t_5=1$; $A_1\equiv(1,\varepsilon,\varepsilon^2,\varepsilon^3)$, $A_2\equiv(1,\varepsilon^2,\varepsilon^4,\varepsilon)$, $A_3\equiv(1,\varepsilon^3,\varepsilon,\varepsilon^4)$, $A_4\equiv(1,\varepsilon^4,\varepsilon^3,\varepsilon^2)$, $A_5\equiv(1,1,1,1)$, $(\varepsilon=\varepsilon_5)$. The throw is invariant for a dihedron group of order 10; two generating permutations are $(A_1\,A_2\,A_3\,A_4\,A_5)$ and $A_1\,(A_2\,A_5)\,(A_3\,A_4)$.

For $n=2$ the other groups are obviously not possible. The results obtained here agree with those of BARRAU.

9. In the following we omit the statement of the coordinates, these being easily calculated by a substitution of the t-values in the equations of the normal curve.

If $n=3$, the throw containing 6 points, we have the following cases.

Cyclic groups.

$m=1$, $k=5$; $t_1=\varepsilon_5$, $t_2=\varepsilon_5^2$, $t_3=\varepsilon_5^3$, $t_4=\varepsilon_5^4$, $t_5=1$, $t_6=0$. The throw is invariant for a cyclic group of order 5, a generating permutation being $(A_1\,A_2\,A_3\,A_4\,A_5)\,A_6$.

If we consider the case $m=2$, $k=3$, we find $t_1=a_1\,\varepsilon_3$, $t_2=a_1\,\varepsilon_3^2$, $t_3=a_1$, $t_4=a_2\,\varepsilon_3$, $t_5=a_2\,\varepsilon_3^2$, $t_6=a_2$. But we can always give a linear transformation of t, viz. $t^*=\left(\dfrac{1}{a_1\,a_2}\right)^{1/2}t$, so that $t_1=p\,\varepsilon_3$, $t_2=p\,\varepsilon_3^2$, $t_3=p$, $t_4=\dfrac{\varepsilon_3}{p}$, $t_5=\dfrac{\varepsilon_3^2}{p}$ $t_6=\dfrac{1}{p}$, which shows that the set is invariant for $t'=\dfrac{1}{t}$ and thus has dihedron symmetry.

$m=3$, $k=2$; $t_1=a_1$, $t_2=-a_1$, $t_3=a_2$, $t_4=-a_2$, $t_5=a_3$, $t_6=-a_3$. The throw is invariant for a cyclic group of order 2, consisting of unity and the permutation $(A_1\,A_2)\,(A_3\,A_4)\,(A_5\,A_6)$.

Dihedron groups.

$k=2$, $m=1$, $m_1=0$, $m_2=1$; $t_1=a$, $t_2=-a$, $t_3=\frac{1}{a}$, $t_4=\frac{-1}{a}$, $t_5=0$, $t_6=\infty$. The throw is invariant for a dihedron group of order 4, the permutations being: $(A_1\,A_2)(A_3\,A_4)(A_5\,A_6)$, $(A_1\,A_3)(A_2\,A_4)(A_5\,A_6)$, $(A_1\,A_4)(A_2\,A_3)(A_5\,A_6)$ and unity.

$k=3$, $m=1$, $m_1=0$, $m_2=0$; $t_1=a$, $t_2=a\,\varepsilon_3$, $t_3=a\,\varepsilon_3^2$, $t_4=\frac{1}{a}$, $t_5=\frac{\varepsilon_3}{a}$, $t_6=\frac{\varepsilon_3^2}{a}$. The throw is invariant for a dihedron group of order 6, the permutations being $(A_1\,A_2\,A_3)(A_4\,A_5\,A_6)$, $(A_1\,A_3\,A_2)(A_4\,A_6\,A_5)$, $(A_1\,A_4)(A_2\,A_6)(A_3\,A_5)$, $(A_1\,A_5)(A_2\,A_4)(A_3\,A_6)$, $(A_1\,A_6)(A_2\,A_5)(A_3\,A_4)$ and unity.

$k=6$, $m=0$, $m_1=1$, $m_2=0$; $t_1=\varepsilon_6$, $t_2=\varepsilon_6^2$, $t_3=\varepsilon_6^3$, $t_4=\varepsilon_6^4$, $t_5=\varepsilon_6^5$, $t_6=1$. The throw is invariant for a dihedron group of order 12; generating permutations being $(A_1\,A_2\,A_3\,A_4\,A_5\,A_6)$ and $A_1\,A_4\,(A_2\,A_6)(A_3\,A_5)$.

In this space we have a throw belonging to the octahedron group, viz. $m=m_1=m_2=0$, $m_3=1$. We have: $t_1=1$, $t_2=i$, $t_3=-1$, $t_4=-i$, $t_5=0$, $t_6=\infty$. The order of the substitution group is 24. The results obtained here for $n=3$ agree with those of BARRAU.

10. We proceed by giving the self-projective throws in S_4. A throw now must contain seven points.

Cyclic groups.

$m=1$, $k=6$, $t_1=\varepsilon_6$, $t_2=\varepsilon_6^2$, $t_3=\varepsilon_6^3$, $t_4=\varepsilon_6^4$, $t_5=\varepsilon_6^5$, $t_6=1$, $t_7=0$. The throw is invariant for a cyclic group of order 6, generated by $(A_1\,A_2\,A_3\,A_4\,A_5\,A_6)\,A_7$.

$m=2$, $k=3$; $t_1=a_1\,\varepsilon_3$, $t_2=a_1\,\varepsilon_3^2$, $t_3=a_1$, $t_4=a_2\,\varepsilon_3$, $t_5=a_2\,\varepsilon_3^2$, $t_6=a_3$, $t_7=0$. The throw is invariant for a cyclic group of order 3, generated by $(A_1\,A_2\,A_3)(A_4\,A_5\,A_6)\,A_7$

$m=3$, $k=2$; $t_1=a_1$, $t_2=-a_1$, $t_3=a_2$, $t_4=-a_2$, $t_5=a_3$, $t_6=-a_3$, $t_7=0$. The throw is invariant for a group of order 2, consisting of $(A_1\,A_2)(A_3\,A_4)(A_5\,A_6)\,A_7$ and unity.

Dihedron groups.

$k=5$, $m=0$, $m_1=1$, $m_2=1$; $t_1=\varepsilon_5$, $t_2=\varepsilon_5^2$, $t_3=\varepsilon_5^3$, $t_4=\varepsilon_5^4$, $t_5=1$, $t_6=0$, $t_7=\infty$. The throw is invariant for a dihedron group of order 10, generated by $(A_1\,A_2\,A_3\,A_4\,A_5)\,A_6\,A_7$ and $A_1\,(A_2\,A_4)(A_3\,A_5)(A_6\,A_7)$.

$k=7$, $m=0$, $m_1=1$, $m_2=0$; $t_1=\varepsilon_7$, $t_2=\varepsilon_7^2, \ldots t_6=\varepsilon_7^6$, $t_7=1$. The throw is invariant for a dihedron group of order 14, generated by $(A_1\,A_2\,A_3\,A_4\,A_5\,A_6\,A_7)$ and $A_1\,(A_2\,A_7)(A_3\,A_6)(A_4\,A_5)$.

In this space obviously no other self-projective throws are possible.

11. As for the self-projective throws in S_n for general value of n, each case must be considered for it self. Indeed the solution depends on the arithmetic properties of the number n. Meanwhile some general remarks can be made. In S_n a throw consists of $n+3$ points. If r_1 is a factor of $n+2$ (which may be the number $n+2$ itself) there clearly always exists a throw, which is invariant for a cyclic group of order r_1. If r_2 is a factor of $n+3$ or of $n+1$, there always exists a throw, invariant for a dihedron group of order $2\,r_2$. In S_5 (more generally: for $n=1, 5, 7, 9, 11$ (mod. 12)) we have throws which are invariant for a tetrahedron group. In S_5 (more generally: for $n=3$, 5, 9, 11, 15, 17, 21, 23 (mod. 24)), we have throws with the symmetry of the octahedron group. The case of a throw, which is invariant for the icosahedron group first occurs in S_9 (and generally for $n=9$, 17, 27, 29, 39, 47, 57, 59 (mod. 60)).

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Mathematics. — *Ueber eine Erweiterung der* LAPLACE-*Transformation*. (Erste Mitteilung). Von C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT).

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§ 1. Einleitung.

Genügt $\varphi(x)$ gewissen Voraussetzungen, so gilt bekanntlich nach den FOURIERschen cos- und sin-Formeln

$$\varphi(x) = \frac{2}{\pi} \int_0^\infty \cos xy \, dy \int_0^\infty \varphi(t) \cos yt \, dt \quad \ldots \ldots \quad (1)$$

und

$$\varphi(x) = \frac{2}{\pi} \int_0^\infty \sin xy \, dy \int_0^\infty \varphi(t) \sin yt \, dt \quad \ldots \ldots \quad (2)$$

Die FOURIERschen Relationen sind sehr nahe verwandt [1]) mit der häufig betrachteten Umkehrformel der LAPLACE-Transformation, d.h. also mit der Formel

$$F(x) = \frac{1}{2\pi i} \int_{\beta-\infty i}^{\beta+\infty i} e^{xs} \, ds \int_0^\infty e^{-st} F(t) \, dt. \quad \ldots \ldots \quad (3)$$

Die Beziehungen (1) und (2) sind Spezialfälle der HANKELschen Formel [2])

$$\varphi(x) = \int_0^\infty J_\nu(xy) (xy)^{\frac{1}{2}} \, dy \int_0^\infty J_\nu(yt) (yt)^{\frac{1}{2}} \varphi(t) \, dt; \quad \ldots \quad (4)$$

---

[1]) Für den Zusammenhang zwischen (1) und (2) einerseits und (3) anderseits vergl. man DOETSCH, [**4**], 89, 94 und 104—105; ferner TITCHMARSH, [**21**], 1—6.

[2]) WATSON, [**22**], 456; BOCHNER, [**1**], 180; TITCHMARSH, [**21**], 240. Viele Arbeiten sind dem Studium der HANKELschen Formel gewidmet worden; ich erwähne hier nur: TITCHMARSH, [**20**]; PLANCHEREL, [**18**]; FOX, [**5**]; OFFORD, [**16**]; OWEN, [**17**]; BRADLEY, [**2**].

denn diese Formel geht wegen

$$J_{-\frac{1}{2}}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\cos z, \qquad J_{\frac{1}{2}}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\sin z$$

für $\nu=-\frac{1}{2}$ in (1) und für $\nu=\frac{1}{2}$ in (2) über.

Das Ziel der vorliegenden Arbeit ist eine mit (4) verwandte Erweiterung von (3) abzuleiten. Mein Hauptresultat lautet wie folgt:

Satz 1. Voraussetzungen: 1. *Die Funktion $F(t)$ sei definiert für $t>0$ und in jedem endlichen Intervall $0<T_1\leqq t\leqq T_2$ im* RIEMANN*schen Sinn eigentlich integrierbar.*

2. *Das Integral*

$$\int_0^\infty e^{-\beta t}\,|F(t)|\,dt$$

sei konvergent für $\beta>a\geqq 0$.

3. *Es sei $x>0$ und $F(t)$ von beschränkter Variation in der Umgebung des Punktes $t=x$.*

Behauptung[3]): *Ist $-\frac{1}{2}\leqq\nu\leqq\frac{1}{2}$ und $\beta>a$, so gilt für den in Voraussetzung 3 genannten Punkt x*

$$\frac{F(x+0)+F(x-0)}{2}=\frac{1}{\pi i}\lim_{\lambda\to\infty}\int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)(xs)^{\frac{1}{2}}\,ds\int_0^\infty K_\nu(st)(st)^{\frac{1}{2}}F(t)\,dt. \tag{5}$$

---

3) Ist $\Re(z)>0$ und $-\frac{1}{2}\leqq\nu\leqq\frac{1}{2}$, so ist $|K_\nu(z)\,z^{\frac{1}{2}}|\leqq\sqrt{\frac{\pi}{2}}\,e^{-\Re(z)}$ (man vergl. WATSON, [22], 219; MEIJER, [8], 658, Satz 1). Aus der zweiten Voraussetzung folgt also, dass das in (5) vorkommende Integral

$$\int_0^\infty K_\nu(st)(st)^{\frac{1}{2}}F(t)\,dt$$

absolut konvergiert.

In vielen Fällen ist

$$\int_{\beta-\infty i}^{\beta+\infty i} I_\nu(xs)(xs)^{\frac{1}{2}}\,ds\int_0^\infty K_\nu(st)(st)^{\frac{1}{2}}F(t)\,dt$$

konvergent; man darf dann bei der Integration nach s den CAUCHYschen Hauptwert

$$\lim_{\lambda\to\infty}\int_{\beta-\lambda i}^{\beta+\lambda i} \quad \text{durch} \quad \int_{\beta-\infty i}^{\beta+\infty i}$$

ersetzen.

Die Funktion $K_\nu(z)$ ist bekanntlich eine gerade Funktion von ν. Aus (5) folgt daher

$$\frac{F(x+0)+F(x-0)}{2}=\frac{1}{2\pi i}\lim_{\lambda\to\infty}\int_{\beta-\lambda i}^{\beta+\lambda i}\{I_\nu(xs)+I_{-\nu}(xs)\}(xs)^{\frac{1}{2}}\,ds\int_0^\infty K_\nu(st)(st)^{\frac{1}{2}}F(t)\,dt. \quad (6)$$

Nun ist

$$I_{\frac{1}{2}}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\sinh z,\ I_{-\frac{1}{2}}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\cosh z,\ K_{\pm\frac{1}{2}}(z)=\left(\frac{\pi}{2z}\right)^{\frac{1}{2}}e^{-z};$$

für $\nu=\pm\frac{1}{2}$ geht (6) also in

$$\frac{F(x+0)+F(x-0)}{2}=\frac{1}{2\pi i}\lim_{\lambda\to\infty}\int_{\beta-\lambda i}^{\beta+\lambda i}e^{xs}\,ds\int_0^\infty e^{-st}F(t)\,dt\ .\ \ . \quad (7)$$

über. Die Beziehungen (5) und (6) sind daher Erweiterungen von (3). Satz 1 ist eine Verallgemeinerung eines DOETSCHschen Satzes [4]).

Falls $F(t)$ stetig ist, lässt Formel (5) sich folgenderweise auseinandernehmen:

$$f(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty K_\nu(st)(st)^{\frac{1}{2}}F(t)\,dt,\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ . \quad (8)$$

$$F(t)=\frac{1}{i\sqrt{2\pi}}\lim_{\lambda\to\infty}\int_{\beta-\lambda i}^{\beta+\lambda i}I_\nu(ts)(ts)^{\frac{1}{2}}f(s)\,ds.\ \ .\ \ .\ \ .\ \ . \quad (9)$$

Mit dieser Bezeichnung besagt Satz 1: Genügt $F(t)$ gewissen Voraussetzungen und wird $f(s)$ durch (8) erklärt, so kann man $F(t)$ mit Hilfe von (9) aus $f(s)$ zurückbekommen.

In den Sätzen 2, 3, und 4 werde ich beweisen: Genügt $f(s)$ gewissen Bedingungen und wird $F(t)$ durch (9) definiert, so gilt (8). Satz 2 ist eine Erweiterung eines Satzes von DOETSCH und CHURCHILL [5]). Ein Spezialfall von Satz 3 kommt bei CHURCHILL vor [6]).

Satz 2. Voraussetzungen: 1. *Die Funktion $f(s)$ sei analytisch in der Halbebene $\Re(s) > a \geqq 0$.*

---

[4]) DOETSCH, [4], 105, Satz 2. Der DOETSCHsche Satz bezieht sich auf Formel (7).

[5]) DOETSCH, [4], 126, Satz 2; CHURCHILL, [3], 569, Satz 1.

[6]) CHURCHILL, [3], 571, Satz 2. Die Sätze von DOETSCH und CHURCHILL beziehen sich auf den LAPLACEschen Fall.

2. *Es sei* $-\frac{1}{2} \leqq \Re(\nu) \leqq \frac{1}{2}$; *für ein festes* $\beta > a$ *und jedes* $t \geqq 0$ *existiere der Grenzwert*

$$\lim_{\lambda \to \infty} \int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(tz)(tz)^{\frac{1}{2}} f(z)\,dz, \quad \ldots \ldots \quad (10)$$

und zwar gleichmässig in jedem endlichen Intervall $0 \leqq t \leqq b$.

3. *Das Integral*

$$\int_{\beta-\infty i}^{\beta+\infty i} \frac{|f(z)|}{|z|} |dz|$$

sei konvergent.

4. *Es sei* $|f(s)| < A$ *für* $\Re(s) \geqq \beta$, *wo* A *eine nicht von* s *abhängige Zahl bedeutet.*

5. *Es sei*

$$\lim_{x \to \infty} f(x + iy) = 0$$

gleichmässig für alle reellen Werte von y.

Behauptung: *Für* $\Re(s) > \beta$ *gilt* (8), *wo* $F(t)$ *durch* (9) *definiert ist.*

Ein Spezialfall von Satz 2 ist

Satz 3. Voraussetzungen: 1. *Die Funktion* $f(s)$ *sei analytisch in der Halbebene* $\Re(s) > a \geqq 0$.

2. *Für ein festes* $\beta > a$ *sei das Integral*

$$\int_{-\infty}^{\infty} |f(\beta + iy)|\,dy$$

konvergent.

3. *Es sei* $|f(s)| < A$ *für* $\Re(s) \geqq \beta$, *wo* A *eine nicht von* s *abhängige Zahl bedeutet.*

4. *Es sei*

$$\lim_{x \to \infty} f(x + iy) = 0$$

gleichmässig für alle reellen Werte von y.

5. *Es sei* $-\frac{1}{2} \leqq \Re(\nu) \leqq \frac{1}{2}$.

Behauptung: *Für* $\Re(s) > \beta$ *gilt* (8), *wo* $F(t)$ *die Funktion*

$$F(t) = \frac{1}{i\sqrt{2\pi}} \int_{\beta-\infty i}^{\beta+\infty i} I_\nu(tz)(tz)^{\frac{1}{2}} f(z)\,dz \quad \ldots \ldots \quad (11)$$

bezeichnet.

Dieser Satz kann sehr leicht aus Satz 2 abgeleitet werden. Man hat nämlich, falls $\Re(\nu) \geqq -\frac{1}{2}$ ist, (man vergl. (14), (15), und (20))

$$|I_\nu(z)\, z^{\frac{1}{2}}| < B\, e^{|\Re(z)|}, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

wo B unabhängig von z ist. Für $0 \leqq t \leqq b$ und $z = \beta + iy$ mit $\beta > 0$ gilt also

$$|I_\nu(tz)\,(tz)^{\frac{1}{2}} f(z)| < B e^{b\beta}\, |f(\beta + iy)|.$$

Man sieht nun sofort ein, dass die Voraussetzungen 2 und 3 von Satz 2 erfüllt sind und dass Satz 3 aus Satz 2 folgt.

Ist $f(s)$ analytisch in der Halbebene $\Re(s) > a \geqq 0$ und $s^k f(s)$ $(k > 1)$ beschränkt für $\Re(s) > a$, so sind die Voraussetzungen von Satz 3 erfüllt, und zwar für jedes $\beta > a$; die durch (11) definierte Funktion $F(t)$ hängt dann wegen des CAUCHYschen Satzes nicht von β ab (man vergl. (12)) und genügt der Integralgleichung (8).

Ist $\Re(\theta) < 0$, $t > 0$ und $\beta > 0$, so gilt bekanntlich [7])

$$\frac{1}{i\sqrt{2\pi}} \int\limits_{\beta-\infty i}^{\beta+\infty i} I_\nu(ts)\,(ts)^{\frac{1}{2}}\, s^\theta\, ds = \frac{2^{\theta+1}\sqrt{\pi}\; t^{-\theta-1}}{\Gamma(\frac{1}{4}+\frac{1}{2}\nu-\frac{1}{2}\theta)\,\Gamma(\frac{1}{4}-\frac{1}{2}\nu-\frac{1}{2}\theta)};$$

ferner hat man [8]), falls $\Re(\frac{1}{2} \pm \nu - \theta) > 0$ und $\Re(s) > 0$ ist,

$$\sqrt{\frac{2}{\pi}} \int\limits_0^\infty K_\nu(st)\,(st)^{\frac{1}{2}}\, \frac{2^{\theta+1}\sqrt{\pi}\; t^{-\theta-1}\, dt}{\Gamma(\frac{1}{4}+\frac{1}{2}\nu-\frac{1}{2}\theta)\,\Gamma(\frac{1}{4}-\frac{1}{2}\nu-\frac{1}{2}\theta)} = s^\theta.$$

Diese zwei Beziehungen sind nur Spezialfälle der Relationen (9) und (8), und zwar mit

$$f(s) = s^\theta, \quad F(t) = \frac{2^{\theta+1}\sqrt{\pi}\; t^{-\theta-1}}{\Gamma(\frac{1}{4}+\frac{1}{2}\nu-\frac{1}{2}\theta)\,\Gamma(\frac{1}{4}-\frac{1}{2}\nu-\frac{1}{2}\theta)}.$$

Die Funktion $f(s) = s^\theta$ genügt aber für $-1 \leqq \Re(\theta) < 0$ weder den Voraussetzungen von Satz 3, noch den Voraussetzungen von Satz 2 [9]). Für diese spezielle Funktion $f(s)$ zeigt also *die direkte Berechnung*, dass die durch (9) definierte Funktion $F(t)$ eine Lösung der Integralgleichung (8) ist.

Ich kann jetzt die folgende Erweiterung von Satz 2 aussprechen:

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[7]) Man vergl. MEIJER, [10], 873, Formel (3).

[8]) WATSON, [22], 388, Formel (8).

[9]) Ist $f(z) = z^\theta$ und $-1 \leqq \Re(\theta) < 0$, so existiert der Grenzwert (10), jedoch nicht gleichmässig in t im Intervall $0 \leqq t < \varepsilon$.

Satz 4. *Genügt $f(s)$ den Voraussetzungen von Satz 2 und wird $g(s)$ durch*

$$g(s) = f(s) + \sum_{h=1}^{n} c_h \, s^{\theta_h}$$

definiert, wo $\Re(\theta_h) < 0 \quad (h = 1, \ldots, n)$, so hat man für $\Re(s) > \beta$

$$g(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty K_\nu(st)\,(st)^{\frac{1}{2}}\, G(t)\, dt,$$

wo $G(t)$ die Funktion

$$G(t) = \frac{1}{i\sqrt{2\pi}} \lim_{\lambda \to \infty} \int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(ts)\,(ts)^{\frac{1}{2}}\, g(s)\, ds$$

bezeichnet.

Es ist klar, dass Satz 3 eine analoge Erweiterung besitzt. Ein verwandter Satz für die LAPLACE-Transformation kommt bei TAMARKIN vor [10]).

§ 2. Hilfssätze.

Ich gebe zunächst einige Hilfsformeln, die ich im Folgenden benutzen werde.

Für die Funktion $K_\nu(z)$ gilt [11]) für festes positives ε und für $|z| \to \infty$

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \{1 + O(z^{-1})\} \qquad (-\tfrac{3}{2}\pi + \varepsilon < \arg z < \tfrac{3}{2}\pi - \varepsilon). \quad . \quad (13)$$

Die entsprechenden Formeln für $I_\nu(z)$ sind

$$\left.\begin{array}{r} I_\nu(z) = \dfrac{e^z}{(2\pi z)^{\frac{1}{2}}}\{1 + O(z^{-1})\} + \dfrac{e^{-z-(\nu+\frac{1}{2})\pi i}}{(2\pi z)^{\frac{1}{2}}}\{1 + O(z^{-1})\} \\ (-\tfrac{3}{2}\pi + \varepsilon < \arg z < \tfrac{1}{2}\pi - \varepsilon) \end{array}\right\} . \quad . \quad (14)$$

und

$$\left.\begin{array}{r} I_\nu(z) = \dfrac{e^z}{(2\pi z)^{\frac{1}{2}}}\{1 + O(z^{-1})\} + \dfrac{e^{-z+(\nu+\frac{1}{2})\pi i}}{(2\pi z)^{\frac{1}{2}}}\{1 + O(z^{-1})\} \\ (-\tfrac{1}{2}\pi + \varepsilon < \arg z < \tfrac{3}{2}\pi - \varepsilon) \end{array}\right\} . \quad . \quad (15)$$

Die Beziehungen (14) und (15) können leicht mit Hilfe von [12])

$$I_\nu(z) = \frac{e^{-\nu\pi i} K_\nu(z) - K_\nu(z e^{\pi i})}{\pi i} = \frac{K_\nu(z e^{-\pi i}) - e^{\nu\pi i} K_\nu(z)}{\pi i}$$

aus (13) abgeleitet werden.

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10) TAMARKIN, [19], 419. Siehe auch DOETSCH, [4], 128, Satz 3.

11) Man vergl. WATSON, [22], § 7. 23; MEIJER, [8], 658, Satz 1.

12) Man vergl. WATSON, [22], 80, Formel (18).

Schliesslich brauche ich noch [13])

$$\int I_\nu(xs)\, K_\nu(ts)\, s\, ds = \frac{s\{x I_{\nu+1}(xs) K_\nu(ts) + t I_\nu(xs) K_{\nu+1}(ts)\}}{x^2 - t^2} + C \quad . \quad (16)$$

Hilfssatz 1. *Es sei $x > 0$, $y > 0$, $r > 0$. Ich betrachte das Integral*

$$L_1 = \left\{ \int_{-\infty i}^{-ri} + \int_{C_r} + \int_{ri}^{\infty i} \right\} I_\nu(xs)\, I_{\nu+1}(ys)\, ds\,; \quad . \quad . \quad . \quad . \quad (17)$$

hierin (und ebenso in (21)) bezeichnet C_r den auf der rechten Seite der imaginären Achse liegenden Halbkreis mit Mittelpunkt $s=0$ und Radius r (von $-ri$ bis ri durchlaufen).

Behauptung:

$$L_1 = \begin{cases} -2ix^\nu y^{-\nu-1} \sin \nu\pi & \text{für } x < y, \\ -ix^{-1} \sin \nu\pi & \text{für } x = y, \\ 0 & \text{für } x > y. \end{cases} \quad . \quad . \quad . \quad . \quad (18)$$

Beweis. Ist $x > 0$, $y > 0$ und $\Re(\nu) > -1$, so gilt bekanntlich [14])

$$\int_0^{\infty i} I_\nu(xs)\, I_{\nu+1}(ys)\, ds = \begin{cases} -e^{\nu\pi i} x^\nu y^{-\nu-1} & \text{für } x < y, \\ -\frac{1}{2} e^{\nu\pi i} x^{-1} & \text{für } x = y, \\ 0 & \text{für } x > y. \end{cases} \quad . \quad . \quad (19)$$

Die linke Seite dieser Beziehung werde L_2 gesetzt. Mit Rücksicht auf

$$I_\nu(z) = \frac{2^{-\nu} z^\nu}{\Gamma(\nu+1)}\, {}_0F_1(\nu+1; \tfrac{1}{4}z^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

sieht man leicht ein, dass L_1 nicht von r abhängig ist und dass dieses Integral für $\Re(\nu) > -1$ den Wert

$$L_1 = (1 - e^{-2\nu\pi i}) \int_0^{\infty i} I_\nu(xs)\, I_{\nu+1}(ys)\, ds = 2ie^{-\nu\pi i} L_2 \sin \nu\pi$$

besitzt. Formel (18) folgt daher sofort aus (19) [15]).

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13) Watson, [22], 134, Formel (8).

14) Watson, [22], 406, Formel (8).

15) Die Bedingung $\Re(\nu) > -1$ darf bei (18) fortgelassen werden, da der Punkt $s=0$ nicht auf dem Integrationswege des Integrals L_1 liegt.

Hilfssatz 2. *Es sei $x > 0$, $y > 0$, $r > 0$. Ich betrachte das Integral*

$$L_3 = \lim_{\lambda \to \infty} \left\{ \int_{-\lambda i}^{-ri} + \int_{C_r} + \int_{ri}^{\lambda i} \right\} I_\nu (x s) I_{-\nu-1} (y s) \, ds. \quad . \quad . \quad . \quad (21)$$

Behauptung:

$$L_3 = -2 i x^\nu y^{-\nu-1} \sin \nu \pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Beweis. Die Integrale

$$\int_{-\infty i}^{-ri} I_\nu (x s) I_{-\nu-1} (y s) \, ds \text{ und } \int_{ri}^{\infty i} I_\nu (x s) I_{-\nu-1} (y s) \, ds$$

sind zwar für $x \neq y$ konvergent [16]), aber nicht für $x = y$. Das ist der Grund, warum ich in Hilfssatz 2 Hauptwerte eingeführt habe.

Wegen (20) gilt

$$\int_{-\lambda i}^{-ri} I_\nu (x s) I_{-\nu-1} (y s) \, ds = - \int_{ri}^{\lambda i} I_\nu (x s) I_{-\nu-1} (y s) \, ds,$$

so dass

$$L_3 = \int_{C_r} I_\nu (x s) I_{-\nu-1} (y s) \, ds$$

ist, und dieses Integral ist mit Rücksicht auf (20) gleich

$$\frac{2 x^\nu y^{-\nu-1}}{\Gamma(\nu+1) \Gamma(-\nu)} \int_{C_r} {}_0F_1 (\nu + 1 ; \tfrac{1}{4} x^2 s^2) \, . \, {}_0F_1 (-\nu ; \tfrac{1}{4} y^2 s^2) \frac{ds}{s}$$

$$= \frac{2 x^\nu y^{-\nu-1}}{\Gamma(\nu+1) \Gamma(-\nu)} \int_{C_r} \frac{ds}{s} = \frac{2 \pi i x^\nu y^{-\nu-1}}{\Gamma(\nu+1) \Gamma(-\nu)} = -2 i x^\nu y^{-\nu-1} \sin \nu \pi,$$

womit der Beweis geliefert ist.

Hilfssatz 3. *Es sei $x > 0$, $y > 0$, $\beta > 0$. Ich betrachte das Integral*

$$L_4 = \lim_{\lambda \to \infty} \int_{\beta - \lambda i}^{\beta + \lambda i} I_\nu (xs) K_{\nu+1} (ys) \, ds. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

---

16) Man vergl. (14) und (15).

Behauptung:

$$L_4 = \left\{ \begin{array}{rl} 0 & \textit{für } x < y, \\ \frac{1}{2}\pi i\, x^{-1} & \textit{für } x = y, \\ \pi i\, x^{\nu}\, y^{-\nu-1} & \textit{für } x > y. \end{array} \right\} \quad \ldots\ldots \quad (24)$$

Beweis. Aus (15) und (13) bezw. (14) und (13) geht hervor, dass die Integrale

$$\int_{\lambda i}^{\beta+\lambda i} I_\nu(xs)\, K_{\nu+1}(ys)\, ds \quad \text{und} \quad \int_{-\lambda i}^{\beta-\lambda i} I_\nu(xs)\, K_{\nu+1}(ys)\, ds$$

gegen Null streben für $\lambda \to \infty$. Das Integral L_4 besitzt daher den Wert

$$L_4 = \lim_{\lambda \to \infty} \left\{ \int_{-\lambda i}^{-ri} + \int_{C_r} + \int_{ri}^{\lambda i} \right\} I_\nu(xs)\, K_{\nu+1}(ys)\, ds. \quad \ldots \quad (25)$$

Für nicht-ganzzahlige Werte von ν gilt aber

$$K_{\nu+1}(z) = \frac{\pi}{2 \sin \nu\pi} \{ I_{\nu+1}(z) - I_{-\nu-1}(z) \};$$

aus (25) ergibt sich also mit Rücksicht auf (17) und (21)

$$L_4 = \frac{\pi}{2 \sin \nu\pi} \{ L_1 - L_3 \}.$$

Formel (24) folgt nun sofort aus (18) und (22) [17].

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[17]) Durch Grenzübergang findet man, dass Formel (24) auch gilt für ganze Werte von ν.

Mathematics. — *On the thermo-hydrodynamics of perfectly perfect fluids.* II. By D. VAN DANTZIG.

(Communicated at the meeting of April 27, 1940.)

Summary.

In § 5 the equations for chemical reactions are given; the mass-action law is obtained as a special case of the condition of isentropy. In § 6 the equations of continuity for the different components in the absence of reactions are introduced. A number of errors often made in relativistic hydro- and thermodynamics is pointed out. The equations of motion and of continuity together are brought in a simple polydimensional (§ 7) and a homogeneous form (§ 8).

§ 5. *Chemical reactions.*

From the equations of motion, e.g. in the form I (25)[28]) follows with $\mathfrak{S}^h = \mathfrak{s}\,\vartheta^h$, $\mathfrak{N}_r^h = \mathfrak{p}_r\,\vartheta^h$, $\mathfrak{s} = -\mathfrak{p}_i\,\vartheta^i - \mathfrak{p}_r\,\lambda^r$:

$$d_{\mathrm{L}}\,\mathfrak{s} = -\vartheta^i\,d_{\mathrm{L}}\,\mathfrak{p}_i - \mathfrak{p}_r\,d_{\mathrm{L}}\,\lambda^r - \lambda^r\,d_{\mathrm{L}}\,\mathfrak{p}_r = -\lambda^r\,d_{\mathrm{L}}\,\mathfrak{p}_r, \quad . \quad . \quad (66)$$

i.e.

$$\partial_j\,\mathfrak{S}^j = -\lambda^r\,\partial_j\,\mathfrak{N}_r^j. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

If the element $\mathrm{d}\,V$ is "dragged along", i.e. if $d_L\,\mathrm{d}\,\mathfrak{V}_i = 0$, this is equivalent with

$$\frac{d}{d\theta}\,S^{\mathrm{d}V} = -\lambda^r\,\frac{d}{d\theta}\,N_r^{\mathrm{d}V}. \quad . \quad . \quad . \quad . \quad . \quad (68)$$ [29])

This equation shows (always for a perfectly perfect fluid!) that *increase of entropy is only possible if the amount of at least one component changes.* This is of course to be expected because of the definition of perfectly perfectness.

Let us suppose now that k chemical reactions can take place between the n components Γ^r, the stoichiometric equations of which are

$$'a_r^u\,\Gamma^r \rightarrow {}''a_r^u\,\Gamma^r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

Here $'a_r^u$ and $''a_r^u$ are non-negative integers, whereas Γ^r are the

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28) Cf. the first part of the paper, these Proceedings **43** (1940), 387—402, referred to as I.

29) The reference on p. 399, I line 6 should be read: cf. § 6, (81).

stoichiometric formulae of the components; $u = I, II, \ldots, (k)$ [30]). Defining $a_r^u =_{\mathrm{df}} {}''a_r^u - {}'a_r^u$, the equations of the reactions become [31])

$$0 \rightarrow a_r^u \Gamma^r \,. \qquad (70)$$

The conservation-law for the charge states that

$$a_r^u e^r = 0, \qquad (71)$$

e^r being the charge of one particle of the r^{th} component. The corresponding relation for the (proper) masses m^r, viz. $a_r^u m^r = 0$ holds only approximately, under neglect of the relativistic variability of the masses. The analogous relation for the non-invariant masses is taken account of already in the energy-law, i.e. in the equations of motion.

The conservation-law for the elementary particles states

$$d N_r^{\mathrm{d}V} = a_r^u R_u^{\mathrm{d}U}, \qquad (72)$$

where

$$R_u^{\mathrm{d}U} =_{\mathrm{df}} \mathfrak{R}_u \, \mathrm{d}\mathfrak{U} \qquad (73)$$

is the number of reactions of type u taking place in a space-time-volume $\mathrm{d}\mathfrak{U} = \mathrm{d}\mathfrak{B}_h \, dx^h$. We suppose here (as usually is the case) that the quantities $\mathfrak{R}_u$, which may depend on the ϑ^h, λ^r and x^h, are functions of these variables alone. Equations (72) are equivalent with

$$\partial_j \mathfrak{N}_r^j = a_r^u \mathfrak{R}_u \,. \qquad (74)$$

By (67), (68) and (72), (74) we obtain:

$$d S^{\mathrm{d}V} = -\lambda^r a_r^u R_u^{\mathrm{d}U} \quad \text{or} \quad \partial_j \mathfrak{S}^j = -\lambda^r a_r^u \mathfrak{R}_u \qquad (75)$$

The condition for isentropy ("reversibility"), viz $d S^{\mathrm{d}V} = 0$ is certainly satisfied if

$$a_r^u \lambda^r = 0. \qquad (76)$$

If only one reaction can take place, we may drop the suffix r; in this case (76) is also *necessary* for isentropy. Condition (76) contains as a particular case the law of GULDBERG and WAAGE. In fact, if every component behaves like an ideal MAXWELL—BOLTZMANN gas (which approximately is the case in very dilute solutions of neutral particles as well as in gasreactions), the partial pressure of the r^{th} component is $\mathfrak{p}_r$. Then

$$\log \mathfrak{p}_r = \lambda^r + \varkappa^r, \qquad (77)$$

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30) E. g. the reaction $2\,HJ \rightarrow H_2 + J_2$ can be written as $0 \rightarrow -2\,\Gamma^I + \Gamma^{II} + \Gamma^{III}$ with $\Gamma^I = HJ$, $\Gamma^{II} = H_2$, $\Gamma^{III} = J_2$, ${}'a_I = -a_I = 2$, ${}''a_{II} = a_{II} = {}''a_{III} = a_{III} = 1$, ${}''a_I = {}'a_{II} = {}'a_{III} = 0$.
Different "components" Γ^r, Γ^s may also be different phases of a single chemical substance, e.g. $\Gamma^I = He_{\mathrm{liq}}$, $\Gamma^{II} = He_{\mathrm{gas}}$. In that case $\lambda^r = \lambda^s$.

31) This method of treatment was introduced by TH. DE DONDER [2].

where $\varkappa^r$ is a function of the temperature, or more generally of the ϑ^h and the x^h, but is independent of the λ^r, so that (76) becomes

$$\Sigma r\, a_r^u \log \mathfrak{p}_r = a_r^u \varkappa^r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

The reaction constant $K_{(u)}$ (which also may depend on ϑ^h and x^h) is determined by

$$\log K_{(u)} = k^u, \quad k^u =_{\mathrm{df}} a_r^u \varkappa^r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

As the $\mathfrak{p}_r$ are proportional with the $N_r^{\mathrm{d}V}$ and with the concentrations, (78) is GULDBERG and WAAGE's mass-action law. The values of the reaction constants are found in the well-known way from the known values of the $\varkappa^r$.

The values of the *reaction-velocities* depend upon the way in which the concentrations are measured (mass-concentrations, volume-concentrations, etc.). They can easily be found as (non-invariant) functions of the $R_u^{\mathrm{d}U}$.

The quantity $a_r^u \lambda^r$ becomes after multiplication with RT ($R =$ gas-constant pro mol) the quantity introduced by DE DONDER [2] under the name of "affinity" of the u^{th} reaction. Also the "*reaction heat*" $q^u = kT^2 \dfrac{d \log K_{(u)}}{dT}$ of the u^{th} reaction is (in contrast with the specific heats) not an invariant. The corresponding entropy

$$s^u = -\frac{q^u}{kT} = -\vartheta^j \frac{\partial k^u}{\partial \vartheta^j} = a_r^u \vartheta^j \left(\frac{\partial \lambda^r}{\partial \vartheta^j}\right)_{\mathfrak{p}_r} \quad . \quad . \quad . \quad . \quad (80)$$

however is. It is remarkable that the majority of the relations of chemical kinetics can be brought into a form, invariant not only under arbitrary transformations of the space-time coordinates (and independent of metrical geometry), but also under linear transformations of the components Γ^r and of the reactions.

§ 6. *Equations of continuity.*

The equations of motion (17) evidently do not determine the motion completely. Together with the chemical equations (74) however they do, provided the $\mathfrak{R}_u$ (as well as $\mathfrak{p}$) are known as functions of the ϑ^h, λ^r and x^h. In fact, we have then $4+n$ equations for as many unknown functions ϑ^h, λ^r of the x^h. It can be proved that the values of ϑ^h and λ^r can be prescribed arbitrarily along a 3-dimensional space $\varphi = 0$ in space time, provided the latter does not contain the direction of ϑ^h (i.e. of the macroscopic motion) and is not tangent to a certain local quadratic cone (Cf. [40])).

As however our knowledge of the functions $\mathfrak{R}_u$ is rather incomplete

we restrict ourselves further to the case that no chemical reactions occur. Then the complementary equations (72), (74) become

$$d N_r^{\mathrm{d}V} = 0, \quad \text{or} \quad \partial_j \mathfrak{N}_r^j = 0, \quad \text{or} \quad d_{\mathrm{L}} \mathfrak{p}_r = 0 \quad . \quad . \quad . \quad . \quad (81)$$

In the second form they constitute the equation of continuity for every component separately.

As a consequence of (81) we find by (67)

$$d S^{\mathrm{d}V} = 0, \quad \text{or} \quad \partial_j \mathfrak{S}^j = 0, \quad \text{or} \quad d_{\mathrm{L}} \mathfrak{s} = 0 \quad . \quad . \quad . \quad . \quad (82)$$

Hence we have:

Theorem 8. *If the amount of every component of a perfectly perfect mixture, moving according to the equations of motion* (17) *remains constant, the entropy also remains constant.*

Perhaps it might be of use to point out here some errors which have often been made in relativistic or even in classical hydrodynamics. They will be formulated for homogeneous fluids ($n = 1$) in vacuo ($f_i^r = 0$). May the reader and the different authors forgive us this unkind chronicle: it does not diminish their great merits, whereas to know the wrecks on the grounds may help future sailors.

1. By almost all authors the proper energy density ϱ is confused with $\bar{\varrho} =_{\mathrm{df}} mc\,\mathfrak{N}_0$, $\mathfrak{N}_0 c^{-1}$ being the proper particle density. In particular this is the case in connection with SCHWARZSCHILD's condition $\varrho =$ const. for an incompressible fluid in relativity theory. This error was pointed out by EDDINGTON [1] p. 122, who however stated that $\bar{\varrho}$ were equal to $\varrho_0 = -\mathfrak{P}^i_{.i} = \varrho - 4\mathfrak{p}$, which also is erroneous, as SYNGE showed. Cf. SYNGE [1], [2]; D. VAN DANTZIG [1] and R. G. § 9.

2. Sometimes $\varrho + \mathfrak{p}$ and ϱ are confused [32]), though several authors warned against this error.

3. The equation of continuity $\nabla_j \mathfrak{N}^j = 0$ seems to have been confused sometimes [33]) with the equation $\nabla_j \varrho\, i^j = 0$, which in general is not valid.

4. If a pressure-density relation exists, the quantity $\varrho + \mathfrak{p}$ is not necessarily equal to $\bar{\varrho}\ (1 + \int_0^{\mathfrak{p}} \bar{\varrho}^{-1}\, d\,\mathfrak{p})$ [34]).

5. The equation $i^i \nabla_j \mathfrak{P}^j_{.i} = 0$, which is a consequence of the equations

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[32]) E.g. TH. DE DONDER [1], p. 54, formula (196), where $\bar{\mu}$, i.e. our $\varrho + \mathfrak{p}$ is supposed to be the density.

[33]) E.g. H. WEYL [1], p. 239. μ_0 is seen to be our ϱ. In accordance with this and error 1, μ_0 is claimed to be constant for an incompressible fluid on p. 186 and p. 238/239. On p. 169 the equation of continuity is given as $\nabla j\, \mu_0\, i^j = 0$

[34]) Cf. e.g. LAMLA [1], PAULI [1], p. 692. LAMLA assumes implicitly that his quantity k is independent of the temperature.

of motion $\nabla_j \mathfrak{P}^j_{.i} = 0$ is often confused with the equation of continuity $\nabla_j \mathfrak{N}^j = 0$. The firstmentioned relation is equivalent with

$$\mathfrak{N}_0^{-1} (\varrho + \mathfrak{p}) \nabla_j \mathfrak{N}^j + \mathfrak{p}' \, d\eta/ds = 0 \quad (\mathfrak{p}' = \partial \mathfrak{p}/\partial \lambda),$$

and therefore leads only to the equation of continuity *if the fluid is known to move reversibly* [35]. The restriction to reversible motion was made explicitly by EINSTEIN [1] but seems to have been forgotten soon.

An increase of the total entropy as considered by some authors [36] is therefore only possible if the correct equation of continuity $\nabla_j \mathfrak{N}^j = 0$ does *not* hold: it can occur only (as long as the fluid remains homogeneous and relativistically perfect) by creation (for $\lambda < 0$) or annihilation ($\lambda > 0$) of matter, not by increase of the disorderedness of the thermal motion [36]. Increase of the *average entropy pro particle* can occur by *annihilation* of matter only, as long as the abovementioned assumptions, as well as the usual inequalities $\varrho + \mathfrak{p} > 0$, $\mathfrak{N}_0 > 0$ are valid.

6. Often the following meanings of statements like $\bar{\varrho} =$ const. are confused, in classical as well as in relativistic hydrodynamics:

a. $\partial \bar{\varrho}/\partial \mathfrak{p} = 0$, i.e. $\bar{\varrho}$ is independent of the pressure (e.g. for constant ϑ^h and x^h);

b. $\nabla_i \bar{\varrho} = 0$, i.e. $\bar{\varrho}$ is constant in space and time;

c. $d\bar{\varrho}/ds = 0$, i.e. $\bar{\varrho}$ is a constant of the motion;

d. $\nabla_i \bar{\varrho} - i_i \dfrac{d\bar{\varrho}}{ds} = 0$, i.e. $\bar{\varrho}$ is constant in every direction orthogonal to the worldlines. Or in classical hydrodynamics: $\partial_a \bar{\varrho} = 0$ $(a = 1, 2, 3)$, i.e. $\bar{\varrho}$ is constant in space.

Evidently *c*. and *d*. together are equivalent with *b*.; otherwise none of these four relations is a consequence from the other ones.

Only interpretation *a*. is in accordance with the linguistic meaning of the word "incompressible". Moreover it is the only one having the nature of an "equation of state", i.e. of a relation between the thermodynamic quantities, independent of the actual state of motion of the fluid (in mathematical terms: independent of the functional dependence of the ϑ^h and λ^r upon the x^h). Several authors however used *b*., *c*. or *d*., claiming it to be a consequence of the equation of state of an incompressible fluid, which it is not. Interpretation *c*. was taken by SYNGE [1]

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35) Cf. e.g. TH. DE DONDER [1] p. 53, formula (193′); A. S. EDDINGTON [1], p. 117, and also Synges criticism against this place, [1], p. 156, [2], p, 390. Our ϱ and $\varrho + \mathfrak{p}$ correspond with EDDINGTONs ϱ_{00} and ϱ_{000} respectively (p. 122).

His equation (52), i.e. in our notation $i^i \nabla_j \mathfrak{P}^j_{.i} = \nabla_j (\varrho + \mathfrak{p}) i^j - d\mathfrak{p}/ds = 0$ is considered by EINSTEIN [2], p. 35, to be the equation of continuity, "welche von derjenigen der klassischen Mechanik nur um das praktisch verschwindend kleine Glied $d\mathfrak{p}/ds$ abweicht".

Here implicitly error 2 (as well as 1) is made, against which the author warned 7 ines higher.

36) Cf. e.g. TOLMAN [1]. Though TOLMAN does not possess our equation (67), he makes a remark analogous to ours on p. 425.

as a *definition* of incompressibility. This of course is logically correct. It would however imply any gas in equilibrium (with respect to a local inertial system) to be "incompressible".

Analogous interpretations are possible for any statement of the form "X is constant", where X is some thermodynamic quantity.

Also for statements like "X is a function of Y", i.e. $\delta X = 0$ if $\delta Y = 0$ an analogous plurality of interpretations is possible, as well as for other relations between variations of thermodynamic quantities. In particular we have to consider whether the relation holds A. for arbitrary variations of all variables (e.g. ϑ^h, λ^r and x^h); B. for purely thermodynamic variations, i.e. variations of the ϑ^h, λ^r only, under constant x^h; C. for arbitrary virtual displacements δx^h with $\delta \vartheta^\varkappa = \delta x^i \, \partial_i \, \vartheta^\varkappa$; D. for the actual variations in time $\delta X = dX$, etc. As an example we mention the fact that in classical hydrodynamics the incomplete system of 4 equations for 5 unknown functions is completed by a relation between $\mathfrak{p}$ and $\bar{\varrho}$, which is usually called an "equation of state" (interpretation B), whereas it is applied as a relation C (with respect to the spatial coordinates x^a). As an example often the adiabatic expansion of a gas is mentioned, where the "constant" $C = \mathfrak{p}\,\tau^{c_\mathfrak{p}}$ is constant only in the sense c. mentioned above ($dC/d\theta = 0$; interpretation D), whereas it is applied in the form $\frac{1}{\bar{\varrho}} \partial_a \mathfrak{p} = \partial_a F$, $F = \int \frac{d\mathfrak{p}}{\bar{\varrho}} = c_\mathfrak{p} \, \tau^{-1}$, equivalent with $\partial_a C = 0$ (constant of type d; interpretation C. with respect to variations of the spatial coordinates).

With respect to these errors it may be noticeable:

A. that many of them are consequences of the habit of treating thermodynamics and hydrodynamics as independent disciplines, which cannot be done consequently;

B. that many of them concern only very small, though typically relativistic corrections;

C. that in classical hydrodynamics they are used sometimes consciously as approximations only, without investigating the bearing of their validity;

D. that our method of treatment, viz invariant and independent of metrical geometry makes it impossible to commit most of them unconsciously.

Bringing the equations in a form independent of metrical geometry (though in practice Euclidean or Riemannian geometry is used) has therefore an analogous effect like writing them in invariant form (though in practice usually a definite system of coordinates is used), and, more elementarily, making them homogeneous with respect to the physical dimensions (though in practice definite units of length, mass, etc. are used). All these methods prevent one from making certain errors.

§ 7. *Polydimensional form of the equations of motion and of continuity.*

By a slight change of notation the equations of motion and of conti-

nuity together can be brought into a very simple and symmetrical form.

We let the indices $\varkappa$, λ, μ, ν run through the range $1, 2, \ldots, 4+n$, so that they gather the indices h and r etc. into one.

By this procedure space-time is imbedded in an auxiliary $(4+n)$-dimensional space, coordinates in which we denote by ξ^{λ}. In particular we may take $\xi^h = x^h$. All quantities introduced hitherto are supposed to be independent of the ξ^r. Hence $\partial_\lambda = (\partial_i, 0)$, i.e. $\partial_s f = 0$ for any f under consideration [37]).

We further define $\vartheta^{\varkappa} =_{\mathrm{df}} (\vartheta^h, \lambda^r)$, $\mathfrak{p}_{\varkappa} =_{\mathrm{df}} (\mathfrak{p}_i, \mathfrak{p}_r)$, $\mathfrak{P}^{h}_{\cdot\,\lambda} =_{\mathrm{df}} (\mathfrak{P}^{h}_{\cdot\,i}, \mathfrak{N}^h{}_r)$, etc. Then we obtain:

$$P_\lambda^{\mathrm{d}V} = \frac{\partial}{\partial \vartheta^\lambda} Z^{\mathrm{d}V} \quad \ldots \ldots \ldots \quad (83)$$

If $\bar{d}_{\mathrm{L}}$ denotes the LIE-differential, belonging to the vectorfield ϑ^{λ} in the same way as d_{L} belongs to ϑ^h, and defining

$$\bar{\omega}_\lambda =_{\mathrm{df}} -\mathfrak{s}^{-1}\, \mathfrak{p}_\lambda, \quad \ldots \ldots \ldots \quad (84)$$

so that

$$\bar{\omega}_\lambda\, \vartheta^\lambda = 1, \quad \ldots \ldots \ldots \quad (85)$$

$\mathfrak{s} = -\mathfrak{p}_\lambda\, \vartheta^\lambda$ being defined like before by $\mathfrak{S}^h = \mathfrak{s}\, \vartheta^h$, we have:

Theorem 9. The equations of motion and of continuity together are equivalent with the set of equations

$$\boxed{\bar{d}_{\mathrm{L}}\, \mathfrak{p}_\lambda = 0}\,, \quad \ldots \ldots \ldots \quad (86)$$

and also with the system

$$\frac{\bar{d}_{\mathrm{L}}}{d\theta}\, \bar{\omega}_\lambda = 2\,\vartheta^\mu\, \partial_{[\mu}\, \bar{\omega}_{\lambda]} = 0, \text{ [38])} \quad \ldots \ldots \ldots \quad (87)$$

$$\frac{\bar{d}_{\mathrm{L}}}{d\theta}\, \mathfrak{s} = \partial_j\, \mathfrak{s}\, \vartheta^j = 0, \quad \ldots \ldots \ldots \quad (88)$$

with the supplementary conditions $\partial_s \vartheta^{\varkappa} = 0$.

Proof. By definition

$$\frac{\bar{d}_{\mathrm{L}}}{d\theta}\, \mathfrak{p}_\lambda = \vartheta^\mu\, \partial_\mu\, \mathfrak{p}_\lambda + \mathfrak{p}_\mu\, \partial_\lambda\, \vartheta^\mu + \mathfrak{p}_\lambda\, \partial_\mu\, \vartheta^\mu \quad \ldots \ldots \quad (89)$$

Hence for $\lambda = i$:

$$\bar{d}_{\mathrm{L}}\, \mathfrak{p}_i / d\theta = \vartheta^j\, \partial_j\, \mathfrak{p}_i + \mathfrak{p}_j\, \partial_i\, \vartheta^j + \mathfrak{p}_r\, \partial_i\, \lambda^r + \mathfrak{p}_i\, \partial_j\, \vartheta^j = 0,$$

37) In order not to complicate the formulae we have dropped factors, transforming the space-time density $\mathfrak{p}$ into a $(4+n)$-dimensional density, etc. Cf. the factors $\mathfrak{z}$ and ω in J. A. SCHOUTEN and D. van DANTZIG [1], p. 459.

38) In D. VAN DANTZIG [6], p. 646, it was shown that every congruence of paths in coordinate-space-time of an ordinary dynamical system satisfies the equations $dq^\mu\, \partial_{[\mu}\, p_{\lambda]} = 0$ entirely analogous with (88) and (99).

according to (18). Moreover for $\lambda = r$:

$$\bar{d}_{\mathrm{L}}\, \mathfrak{p}_r / d\theta = \vartheta^j\, \partial_j\, \mathfrak{p}_r + \mathfrak{p}_r\, \partial_j\, \vartheta^j = 0$$

according to (81), (20). Hence (86) is proved. Because of $\dot{d}_{\mathrm{L}}\, \vartheta^{\varkappa} = 0$ (cf. (23)), (86) leads to $\bar{d}_{\mathrm{L}}\, \mathfrak{s} = -\vartheta^{\lambda}\, \bar{d}_{\mathrm{L}}\, \mathfrak{p}_{\lambda} = 0$, i.e. to (88). Hence the second part of the theorem follows immediately from (84), (85) and (22).

Transvecting equations (87) with $\delta\xi^{\lambda}$, they are seen to be equivalent with

$$d\,\overline{\omega}_{\lambda}\, \delta\, \xi^{\lambda} - \delta\, \overline{\omega}_{\lambda}\, d\, \xi^{\lambda} = 0. \qquad (90)$$

The left side is the socalled bilinear covariant of the differential form

$$d\,\theta = \overline{\omega}_{\lambda}\, d\, \xi^{\lambda}. \qquad (91)$$

The equation (90) is of the same type as the equations of a mechanical system with coordinates and time $q^{\varkappa}$ and momentum and negative energy p_{λ}, which are equivalent with

$$d\, p_{\lambda}\, \delta\, q^{\lambda} - \delta\, p_{\lambda}\, d\, q^{\lambda} = 0. \qquad (92)$$

The left side of (92) of course is the bilinear covariant of the differential form

$$d\, \Lambda = p_{\lambda}\, d\, q^{\lambda}, \qquad (93)$$

the variation of which vanishes along the paths.

An important difference however is the fact that between the p_{λ} and $q^{\varkappa}$ a relation $\mathcal{H}(p_{\lambda}, q^{\varkappa}) = 0$, equivalent with $p_0 + H(p_{\alpha}, q^{\varkappa}) = 0$ exists, whereas the $\omega_{\lambda}, \xi^{\varkappa}$ are independent thermodynamic variables. Instead of such a relation $\mathcal{H} = 0$ we have here the equation of continuity (88) for the entropy. As this equation is not invariant under contact tranformations leaving invariant the differential form (91), but is essentially bound to a definite congruence of paths, the solution of the complete system of equations (87), (88) can not immediately be reduced to the solution of a HAMELTONian system.

§ 8. *Homogeneous variables.*

A still more symmetrical formalism is obtained if we introduce homogeneous variables. Therefore we introduce a new coordinate ξ^0 and a new independent variable (function of the coordinates) Θ^0. We define $\Theta^{\varkappa} =_{\mathrm{df}} \Theta^0\, \vartheta^{\varkappa}$ and let the suffixes A, B, C, D, E run through the range $0, 1, 2, 3, 4, \ldots 4+n$. Substituting $\vartheta^{\varkappa} = \Theta^{\varkappa}/\Theta^0$ into $\mathfrak{p}$, $\mathfrak{p}$ becomes a function of the Θ^A and we may build

$$\Pi_B =_{\mathrm{df}} \mathfrak{p}^{-1} \frac{\partial\, \mathfrak{p}}{\partial\, \Theta^B}, \qquad (94)$$

whence

$$\Pi_0 = (\mathfrak{p}\, \Theta^0)^{-1}\, \mathfrak{s}, \quad \Pi_{\lambda} = (\mathfrak{p}\, \Theta^0)^{-1}\, \mathfrak{p}_{\lambda} = -\overline{\omega}_{\lambda}\, \Pi_0.$$

As $\mathfrak{p}$ is homogeneous of degree zero with respect to the Θ^A, EULER's theorem gives

$$\Pi_A \, \Theta^A = 0. \quad \ldots \ldots \ldots \quad (95)$$

Hence $\delta \log \mathfrak{p} = \Pi_B \, \delta \, \Theta^B = - \, \Theta^B \, \delta \, \Pi_B$, so that

$$\Theta^A = \mathfrak{p} \frac{\partial \, \mathfrak{p}^{-1}}{\partial \, \Pi_A}. \quad \ldots \ldots \ldots \quad (96)$$

Hence there is a complete duality between the Π_B and the Θ^A. Evidently [39])

$$\Pi_B :: (-1, \overline{\omega}_\lambda) :: (\mathfrak{s}, \mathfrak{p}_i, \mathfrak{p}_r);$$
$$\Theta^A :: (1, \vartheta^h, \lambda^r) :: (d\,\theta, dx^h, d\xi^r) :: (kT, u^a, 1, -\mu^r),$$

where $a = 1, 2, 3$, $u^a = \dfrac{dx^a}{dt}$ being the ordinary velocity and $\mu^r = -\lambda^r \, kT$ the ordinary chemical potentials.

Defining (cf. [5])):

$$\mathfrak{P}^A_{.B} =_{\text{df}} \Theta^A \, \Pi_B + \mathfrak{A}^A_B, \quad \ldots \ldots \ldots \quad (97)$$

we have

$$\left.\begin{aligned} \mathfrak{P}^A_{.B} &= \mathfrak{p}^{-1} \frac{\partial \, \mathfrak{p} \, \Theta^A}{\partial \, \Theta^B} = \mathfrak{p}^{-1} \frac{\partial^2 \, \mathfrak{p}}{\partial \, \Theta^B \, \partial \, \Pi_A} = \\ &= \mathfrak{p} \frac{\partial \, \mathfrak{p}^{-1} \, \Pi_B}{\partial \, \Pi_A} = \mathfrak{p} \frac{\partial^2 \, \mathfrak{p}^{-1}}{\partial \, \Pi_A \, \partial \, \Theta^B}. \end{aligned}\right\} \quad \ldots \ldots \quad (98)$$

Moreover by (95)

$$\det (\mathfrak{P}^A_{.B}) = 1.$$

If $d\mathfrak{X} = d\xi^0 \, d\xi^5 \ldots d\xi^{4+n}$ [37]), $d\mathfrak{W}_i = d\mathfrak{V}_i \, d\mathfrak{X}$, $d\mathfrak{W}_0 = d\mathfrak{W}_r = 0$, then we have evidently

$$\mathfrak{P}^A_{.B} \, d\mathfrak{W}_A = \mathfrak{p}^{-1} \, d\mathfrak{X} \, (S^{dV}, P^{dV}_b, -E^{dV}, N^{dV}_r).$$

Defining

$$\left.\begin{aligned} \Pi_{CB} &=_{\text{df}} \frac{\partial \, \Pi_B}{\partial \, \Pi^C} = \frac{\partial^2 \log \mathfrak{p}}{\partial \, \Theta^C \, \partial \, \Theta^B}, \\ \Theta^{BA} &=_{\text{df}} \frac{\partial \, \Theta^A}{\partial \, \Pi_B} = \frac{\partial^2 \log \mathfrak{p}^{-1}}{\partial \, \Pi_B \, \partial \, \Pi_A}, \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (99)$$

we see that both tensors are symmetrical and non degenerate, as

$$\Theta^{AB} \, \Pi_{BC} = \mathfrak{A}^A_C \quad \ldots \ldots \ldots \quad (100)$$

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[39]) :: means "is proportional with".

Moreover, Π_B being homogeneous of degree -1 with respect to the Θ^A and vice versa,

$$\left.\begin{array}{c} \Pi_{BC}\,\Theta^C = -\,\Pi_B, \quad \Theta^{AB}\,\Pi_B = -\,\Theta^A, \\ \Pi_{AB}\,\Theta^A\,\Theta^B = \Theta^{AB}\,\Pi_B\,\Pi_C = 0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (101)$$

We define the operator ∂_0 to annihilate all quantities introduced until this paragraph, in particular therefore the ratio's of the Θ^A as well as of the Π_B and the products $\Pi_B\,\Theta^A$; $\partial_B\,\Theta^0$ shall be defined by requiring

$$\boxed{\partial_B\,\mathfrak{p}\,\Theta^B = 0.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

Then an easy calculation shows the validity of

Theorem 10. *Assumed the definitions and restrictions mentioned above, the equations of motion and of continuity are equivalent with the system consisting of* (102) *and*

$$\boxed{\Theta^B\,\partial_{[B}\,\Pi_{A]} = 0} \quad ^{38)}\ ^{40)} \quad . \quad . \quad . \quad . \quad . \quad (103)$$

Analogous remarks as those made at the end of § 7 with respect to equations (87), (88) evidently are valid with respect to (103), (102).

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40) It follows that the quadratic cone mentioned at the beginning of § 6 is

$$(\Theta^{ij} - \Theta^i\,\Theta^j)\,\varphi_i\,\varphi_j = 0, \quad \varphi_i = \partial_i\,\varphi, \quad \partial_0\,\varphi = \partial_r\,\varphi = 0.$$

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Mathematics. — *Die Triangulation der differenzierbaren Mannigfaltigkeiten. — Nachtrag.* By HANS FREUDENTHAL. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of April 27, 1940.)

Herr S. S. CAIRNS hat mich auf einige seiner Arbeiten aufmerksam gemacht, die ich leider übersehen hatte. Nach Durchsicht dieser Arbeiten ergänze ich die historische Einleitung zu meiner Note „Die Triangulation der differenzierbaren Mannigfaltigkeiten" wie folgt:

S. S. CAIRNS hat bereits 1935 einen Beweis der Triangulierbarkeit der differenzierbaren Mannigfaltigkeiten veröffentlicht. Die Methoden von L. E. J. BROUWER und von mir sind völlig verschieden von der von CAIRNS.

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1) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 880—901 (1939).

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Geology. — *A contribution to the geology of Bawean.* By F. KEIJZER. (Communicated by Prof. L. RUTTEN.)

(Communicated at the meeting of April 27, 1940.)

The isle of Bawean in the Java-sea, North of Surabaya, has been visited thus far only a few times by geologists. In 1851 the mining-engineer C. DE GROOT paid a visit to the island in order to make an investigation into the value of the coal-bearing strata at the Soengei Radja. Only a preliminary report, accompanied by a geological sketch-map, has been published on this survey (1). In 1875 ZIRKEL described a rock-sample from Bawean as the first leucite-bearing rock found outside Europe (2). A fortnightly survey by VERBEEK in 1886 resulted in a thorough description of his observations (3). The map accompanying his report looks like being very detailed, but in fact many roads in the interior have not been surveyed, resulting in errors in his geological map. Chemical analyses and petrographical descriptions of rocks from Bawean have been published by IDDINGS and MORLEY (4). These analyses also may be found in a recent compilation by WILLEMS (5). Some spectrographical examinations have been made by VAN TONGEREN (6). K. MARTIN described some Lamellibranchiata from the tertiary beds (7, 8).

Thanks to the kindness of professor J. I. J. M. SCHMUTZER of the State University at Utrecht I have been able to study a collection of rock-samples made by him in Bawean in 1912, and now in possession of the Mineralogical-Geological Institute at Utrecht, together with some samples, which in the time have been collected by VERBEEK. Moreover some samples have been kindly put at my disposal by the Colonial Institute at Amsterdam. I must express my thanks to professor SCHMUTZER for very carefully looking through all the thin-sections and for his valuable advise.

All find-spots of the samples, with exception of those received from the Colonial Institute have been indicated on the map in fig. 1.

Petrology.

Phonolites. Collection SCHMUTZER: 17—19, 21—25, 27, 47, 52—54, 58—61, 64, 71—74, 76, 79—84, 95—98, 102, 103, 108, 110—116. Collection VERBEEK: 214.

Dense rocks, generally with a greasy lustre, and greenish, greyish or brownish colours. Often showing distinct flowing-texture. Small phenocrysts, never exceeding 2 to 3 mm in size, of sanidine, nepheline and dark minerals may be recognized with the naked eye. Weathering begins with

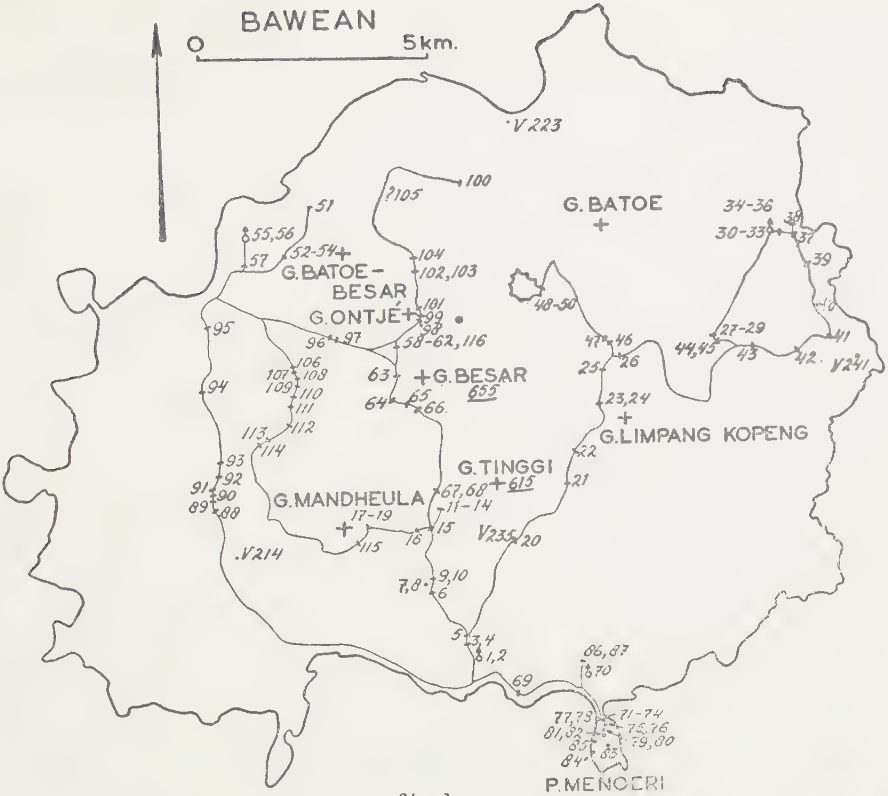

fig.1

fig.2

a : olivine; b : rhönite; c : yellow mineral;
d : plagioclase; e : leucite; f : pyroxene.

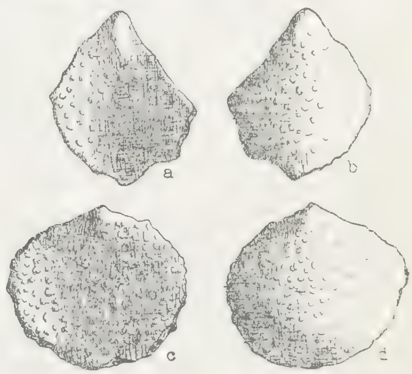

fig.3

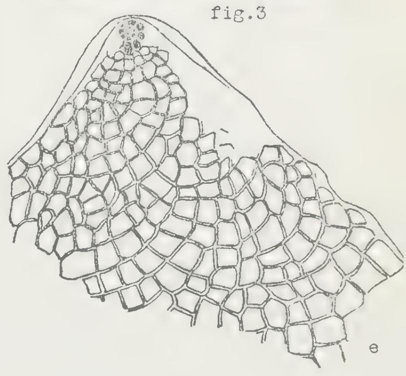

Fig. 1.

the forming of a light-brown to yellow crust and results in yellowish-white, soft and friable rocks.

Under the microscope the rocks show a distinct porphyritical texture. The sanidine is always clear and fresh, and tends to become lathshaped. In V 214 it is definitely narrow lathshaped. Besides the normal claevage-planes a third claevage parallel (100) is developed with undulating planes, as mostly occurring in phonolitic rocks. Karlsbad twins are very common. The phenocrysts often have been rounded off after their cristallization, but later this has been followed by a renewed deposition of sanidine along the borders of the phenocrysts, whereby the demarcation from the groundmass has become indistinct. The phenocrysts may reach a size up to 3 mm.

The nepheline phenocrysts are short, clear prisms. In 60 they contain many small inclusions arranged in planes parallel to (0001). In 18 the nepheline has been replaced partly by a zeolite. The dimensions are always smaller than those of the sanidine, but still may reach 1 to 2 mm.

The pyroxene is a colourless diopside, an aegirineaugite, or an aegirine. These may simultaneously occur in one zonally built crystal; in this case the diopside always forms the centre. While the aegirine may occur as the only representative of the pyroxenes, the diopside is always accompanied by the other members. Twins are rare.

Dark brown biotite as a phenocryst occurs fairly often. Mostly it appears in an aggregate with pyroxene and/or olivine (58, 59, 74, 79) and sometimes hornblende also (84). Hornblende occurs more often than biotite. In almost all rocks it appears as a totally opacitized phenocryst, surrounded by small pyroxene crystals. The unaltered hornblende appears to be greenish-brown (79, 80) or green (84, 95, 97). Typical alcali-hornblendes have not been met with. Olivine has been found in 58, 73, 74, 79, 80. Generally it is fresh with a brown rim. In some of the cases it occurs in aggregates with pyroxene and biotite. In 79 it reaches a size of 3 mm. Apatite occurs in practically all samples, sometimes evenly distributed throughout the rock, but mostly enclosed in the dark phenocrists.

The matrix seems to be always holocrystalline. If any glass occurs at all, it plays a very unimportant role. In the nephelinitoid types the matrix consists for the bigger part of typical small nephelines, together with pyroxenes and small ore-grains, while allotriomorphic sanidine is intermixed. When the nepheline-sections are not evident, a nephelinitoid substance with small birefringence is found, which only difficultly can be distinguished from sanidine. Noteworthy is the dominating of the nephelinitoid types. Only 97 is more to the trachytoid side. Here the sanidine is much better idiomorphically developed. The more nephelinitoid the rock-type, the less the quantity of dark minerals it contains. In the extreme nephelinitoid types the dark phenocrysts have almost entirely disappeared and the matrix is full of small aggregates and streaks of aegirine microlites, while sometimes also streaks of colourless diopsidic pyroxene occur (18, 19, 98, 102). Biotite and hornblende are lacking in the second genera-

tion. In the matrix of 72, 73, 74 numerous small turbid crystals of some sodalite-mineral were found.

Leucite-phonolites. (Collection SCHMUTZER: 49—51, 62, 63, 75, 85. These rocks cannot be sharply distinguished from the foregoing pure nepheline-phonolites, as all transitions exist from nepheline-phonolite with only a few grains of leucite (49, 50, 51) to a pure leucite-phonolite, wherein all nepheline has been replaced by leucite (75). The typical representatives are light-grey rocks, rough to the touch, and looking much like the later to be described trachy-andesites. Except by the occurring of round leucite-grains in the matrix and the disappearing of nepheline, they do not differ in any respect from the nepheline-phonolites. Leucite never occurs as a phenocryst. In 49 the pyroxene has a peculiar yellow colour, probably due to the influence of fumaroles. Melanite has been found in 62 as a large greenishbrown crystal with small inclusions of ? zircone. Titanite occurs as a large irregular phenocryst in 51.

Nephelinites. Collection SCHMUTZER: 28, 99.

Dark-grey, dense rocks, with small phenocrysts of dark minerals. Under the microscope 99 shows many, partly opacitized brownish-green hornblende phenocrysts, zonally built diopside and aegirine-augite, and brown biotite. The matrix totally consists of allotriomorphic nepheline, many small augites and ore-grains. Apatite is enclosed in the phenocrysts. The thin-section contains many patches of secondary calcite.

In 28 there is still a little sanidine in the matrix. Hornblende is lacking, but pyroxene with zonal structure is abundant. Less biotite than in 99, while apatite occurs as small dusty prisms in the matrix and in the phenocrysts.

Trachy-andesites. Coll. SCHMUTZER: 106, 107, 109.

These rocks show white and reddish felspar phenocrysts, up to some mm, and smaller dark phenocrysts in a rough, light-grey to yellow-grey matrix. Under the microscope the three rocks resemble each other closely. The basic plagioclase phenocrysts are always fresh and beautifully twinned according to the albite-law and sometimes also to the pericline-law. The dark phenocrysts are an opacitized hornblende and a green diopsidic augite. The latter is often zonally built with a colourless centre. Apatite occurs in small dusty prisms near the pyroxenes and in very small crystals in the matrix. The matrix itself consists mainly of sanidine, twinned according to the Karlsbad-law, a little plagioclase, augite-microlites and small octaedres and grains of magnetite.

Leucite-tephrites. Coll. SCHMUTZER: 3, 5, 6, 10, 13, 14, 16, 26, 38—40, 43—45, 66, 67, 69, 77, 88, 89, 91—93, 104. Coll. VERBEEK: 241.

Light- to dark-grey or black, very fine grained or dense rocks with many phenocrysts of pyroxene. Some of the samples show oblong gas-cavities in the black matrix (66). Sometimes the matrix is greyish-purple (67).

Weathered rocks have a beige-brown crust, in which the black augites are sharply outlined. Under the microscope always distinct porphyritical rocks, in which the main part of the phenocrysts is formed by light gray-green and colourless diopsidic augite, sometimes with the characteristic violet colour indicating a Ti-content. In the second place comes brown biotite, which occurs almost in all samples, except in 5 and 13 and in the group 88, 89 and 91—93. This biotite always shows resorption phenomena, viz.: a replacement by small ore-grains and augite-microlites or a replacement by rhönite. Both processes may occur simultaneously. VERBEEK already mentions this rhönitization, takes it however for rutile. The rhönite-laths are pleochroitic (dark brown to black) and penetrate from the periphery along the cleavage-planes so that the process is best to be seen in sections, parallel to (001), in which the rhönite-laths seem to intersect at angles of 60°. In most cases big biotite individuals remain unaltered in the centre. Especially well to be seen is the rhönitization in 14, 38, 45, 66, 67, 77 and V 241. Only biotite has been affected by rhönitization. When biotite is enclosed by pyroxene or hornblende it remains unaffected.

Hornblende is common, but mostly totally opacitized and surrounded by augite-microlites. When it is unaltered in the centre it appears to be a brown (10, 16) or a green variety (38). The group of 88, 89, 91—93 is characterized by abundant opacitized hornblende phenocrysts and by the lacking of biotite. Moreover in these rocks haüyne or nosean is found. The "nosean" is too small to be perceived by the naked eye. It has a colourless centre and a brown rim, while two systems of lines of inclusions are cutting each other rectangularly. Blue haüyne is only found in small crystals in 16 and 26. The matrix consists mainly of small, clear leucites and plagioclase-laths of microscopical dimensions. The small leucites only rarely show the characteristic concentrical rings of inclusions, and only the somewhat bigger individuals show birefringence. Small augites occur mostly in considerable quantities, just as small ore-grains, which are, together with the augites, responsible for the dark colour of the rock.

Leucite-basanites. Coll. SCHMUTZER: 4, 11, 12, 34—46, 39x, 41, 68. Coll. VERBEEK: V 235.

These rocks only differ from the leucite-tephrites by the presence of olivine among the phenocrysts and sometimes in the matrix (4,39x). Augite and biotite, the latter having been affected by the same resorption-processes as already mentioned, are the same as in the tephrites. Hornblende, haüyne and nosean are lacking. The olivine has been subject to two processes: 1) an alteration beginning with the deposition of carbonates upon the cracks in the olivine and resulting in a total replacement of the olivine by a mixture of carbonates, opal and chalcedone, 2) a replacement by iddingsite. These alterations never occur simultaneously. One rock deserves a more detailed description:

4: This rock is especially remarkable by the intense iddingsitization of

the olivine, which occurs both as a phenocryst and as a component of the matrix. The olivine in the matrix has been replaced for the bigger part by beautifully orange-red iddingsite. The phenocrysts only have been affected at the rim, where, just as in the small individuals, the fibrous structure of the iddingsite is evident. Striking, however, is the surrounding of the olivine phenocrysts by an aggregate of rhönite and a yellow, roughly lath-shaped mineral. This mineral shows a rather strong birefringence, straight extinction, negative zone-character; it is not pleochroitic. The optical character seems to be uni-axial negative or bi-axial negative with a small optical angle. It is probably some modification of iddingsite, from which it is very difficult to distinguish. The small dimensions however prevent the obtaining of more details. The spaces between the unknown, yellow mineral and the rhönite are filled up by plagioclaselaths and leucite (fig. 2, × 50). The whole aggregate gives the impression of being a reaction-rim around the olivine.

Leucitites. Coll SCHMUTZER: 29, 42.

Olivine-leucitites. Coll. SCHMUTZER: 9, 37, 65, 94. Coll. VERBEEK: V 223.

There is no sharp boundary between the two types. Both are dark, almost black, fine-grained rocks with augite-phenocrysts and some olivines, while in the olivine-leucitites the latter are more abundant. The leucitites are somewhat lighter coloured. The leucitites show phenocrysts of augite — the same as in the basanites and tephrites — and biotite, which is mostly partly replaced by fine ore-grains, in a matrix rich in leucite. Through the abundance of sanidine-cement and the perfect clearness of the leucites the crystal-forms of the latter have become indistinct. There is much second-generation augite, but relatively few ore.

The olivine-leucitites are characterized by the lacking of the sanidine-cement and the presence of olivine in the matrix. Between the two groups all transitions exist. In the leucitites a few olivine phenocrysts may be present, and in the olivine-leucitite 37 f.i. a little sanidine-cement is found in the matrix. The olivine-phenocrysts are mostly strongly corroded, have a yellow-brown rim and are generally fresh. Ore is more abundant than in the leucitites. In 65 again rhönite is found together with a yellow mineral surrounding olivine, just as in no. 4 among the basanites. Here, however, the olivine is not iddingsitized, and the phenomenon is much less evident. In 37 a biotite phenocryst is rhönitized: small feather-shaped crystal-skeletons of rhönite are lying in the cleavage-planes. This is to be seen in more cases, but it is nowhere as distinct as here. By the presence of plagioclase-laths in the matrix (V 223) these rocks grade into the leucite-basanites.

Tuffs. Collection SCHMUTZER: 15, 20, 46, 48, 78, 83, 90.

Very different types are represented.

15: Very probably a tuff, but its character could not be determined any

further. It is a light-grey, friable rock, which shows under the microscope a cloudy matrix, wherein many, very small, clear mineral-particles are imbedded, the true nature of which could not be determined, owing to the small dimensions.

20, 46, 48: These rocks are remarkable by their content of quartz. 20 shows a dusty green matrix with many augite-fragments, biotite — mainly replaced by ore-grains — and clear, angular quartz-fragments. 46 contains small phonolite-fragments, fragments of diopsidic augite and of biotite and very many angular quartz grains. The latter sometimes show wavy extinction. A few glauconite-grains are present, and some fragments of an organogeneous limestone. This rock is evidently a *tuffite*. 48 contains many fragments of leucite-tephrite, less of phonolite, loose fragments of augite, biotite and quartz in a dusty matrix with speckles of ore and very small mineralfragments. The quartz-grains sometimes have a slightly wavy extinction. The numbers 20 and 48 may be tuffites, just as 46, but here are other possibilities also. 20 is found quite near an exposure of tertiary sedimentary rocks, which always have a content of quartz-grains, so that it might be a tuff mixed up with components of those rocks. This possibility does not exist for the sample 48, that was taken at the shore of the crater-lake, but here (and this also holds good for the other two) the quartz-grains may have been derived from a quartz-bearing substratum. In consequence of the explosive origin of the tuffs, time may not have been long enough for melting the quartz. Personally I would take 20 for a tuffite and 48 for a tuff.

78: An altered tephrite-tuff. Many green augites and augite-fragments are imbedded in a fine matrix with still well recognizable felspar-laths. Biotite has been changed into ore-grains.

83: is a very weathered rock, light beige-brown and very soft. Possibly, according to the structure in thin-section, an altered phonolite or phonolite-tuff.

90: Is a totally altered tephrite-tuff. Only the sections of leucites are evident. Augite has been totally limonitized.

101 consists of angular fragments of volcanic rocks and looks like a fine breccia. Phonolites, leucitic and tuffaceous rocks are represented among the components. The cement still contains many very small fragments and has a content of calcite. A cavity is filled up by zeolites. This rock may as well be a coarse tuff or a younger breccia.

Deposits of hot springs. Coll. SCHMUTZER: 1, 2, 30—33, 55, 56, 70. Either calcareous tufa or coarsely crystalline limestones.

Tertiary sedimentary rocks.

Collection SCHMUTZER: 7, 8, 57, 86, 87, 100, 105: collection Colonial Institute: Java 35, 37, 43.

7 is a pure yellow-grey quartzsandstone, resting on number 8;

8 is a dense, beige limestone, rich in Foraminifera, of which the following genera and species could be determined:

Lepidocyclina (Nephr.) sumatrensis BRADY var. inornata RUTTEN;
Miogypsina primitiva TAN;
Trillina howchini SCHLUMB.;
Alveolinella bontangensis RUTTEN;
Cycloclypeus cf. carpenteri BRADY;
Gypsina howchini CHAPMAN.
? Sorites sp.;
Operculina sp.

All these forms are strikingly small. The association proves, that the limestone belongs to the lower parts of "tertiary f".

57 is a snow-white soft limestone, from which with some difficulty a number of foraminifera could be isolated:

Lepidocyclina (Nephr.) sumatrensis BRADY var. inornata RUTTEN; A- and B-form;
Miogypsina primitiva TAN, B-form;
Alveolinella bontangensis RUTTEN;
Gypsina howchini CHAPMAN;
Gypsina globulus REUSS;
Nonion pompiloides FICHTEL & MOLL;
Operculina sp.;
Clavulina sp.;
Clavulinoides sp..
Nodosaria sp.

The preservation is not so good as to allow the determination of more smaller foraminifera. This limestone equally belongs to the lower parts of "tertiary f".

86 is a totally silicified limestone which contains at the side of fragments of corals and casts of lamellibranchiata Lepidocyclina sp. and Miogypsina primitiva TAN.

87 is also a silicified limestone with badly preserved Lepidocyclina sp. and Cycloclypeus sp.

105 is a silicified coral-limestone.

Java 35: Argilaceous sandy limestone with Pecten sp., Soengei Radja. A thin-section shows very abundant and beautifully preserved Foraminifera, but it is impossible to isolate them from the rock. The following Foraminifera were encountered:

Miogypsina sp.;
Cycloclypeus sp.;
Miliolidae;
Operculinidae;
Rotalidae.

Java 37: A hard, dark-brown shale. Tandjoeng Lajar.

Java 43: A sandy concretion with gypsum. Soengei Radja.

Miogypsina primitiva TAN, B-form (fig. 3a, b, c, d, × 10; 3e × 35).

The A-form has been described by TAN in a special paper on Miogypsinidae (9). The B-form had not yet been mentioned. It shows the characteristic wall-structure of the A-form, as figured by TAN (9, page 89, fig. 1, 2). The nepionic spiral is clearly of the complanata-type. The preservation of our specimens however is too bad to allow the study of the position of the stoloniferous apertures.

Gypsina howchini CHAPMAN.

A typically flattened Gypsina, biconvex to concave-convex. Our specimens agree well with those described by CHAPMAN (10).

Alveolinella bontangensis RUTTEN.

In 1938 professor RUTTEN has been informed by Miss I. CRESPIN, palaeontologist of the Commonwealth of Australia, that the Alveolinella bontangensis named by him in 1912 (12), might have been described earlier by CHAPMAN as Alveolina cucumoides (11). CHAPMAN's description however is non-committal and the pictures also are not convincing. Miss CRESPIN it at work on a paper on this subject, but as long as her results have not been published, the name of Alveolinella bontangensis must still be used.

Concerning the age of the Bawean-volcano the following remarks can be made: VERBEEK assumes the volcano to be older than the tertiary sediments, from which follows, that the age would be pre-upper-miocene. For this assumption VERBEEK only supplies one proper argument, namely the occurrence of small fragments of tephrite and phonolite in some marls. Further he writes, that the sediments make the impression of having been deposited upon the volcano. Against his supposition several objections can be made:

1) The volcano still is in a youthful erosion-stage. The strong tropical erosion taken into consideration, it seems impossible, that the volcano-form might have been preserved so well since the upper-miocene. A crater-lake is still present, and there are indications of two different "somma's". The certainly quaternary Ringgit leucite-volcano in Eastern Java is now a strongly dissected volcano-ruin with sharp peaks.

2) VERBEEK assumes, that the present position of the tertiary sediments is the result of a simple central elevation of the body of the volcano. But against this assumption are pleading his own observations of steep dips. In the Soengei Radja complex he mentions 27°, and in his section 16 even 45°, while in the latter section the strike is almost perpendicular to the direction to be expected.

3) In the tertiary sediments — and this holds the more for the sediments at Kadoe kadoe, which according to VERBEEK would have been deposited in a bay — very pure quartz-sandstones occur, without any trace of volcanic components. We must take into consideration, that the fragments of volcanic rocks, mentioned by VERBEEK as occurring in some

marls, may have been washed-in later. While a similar washing-in of foreign elements in a *limestone* would be inconceivable, it is not impossible, that the *marls* mentioned by VERBEEK have been remoulded secondary, and during this remoulding have been contaminated with volcanic components. Moreover, we must point to the fact, that in our samples of tertiary sediments no trace of volcanic elements could be detected.

From the foregoing it will be clear, that the volcano is very probably younger than the tertiary deposits, though direct proof, as contact-metamorphism or superposition of the volcanic rocks upon the tertiary sediments is still lacking. As our number 46 is certainly a tuffite with marine components, it is evident, that the sea must have reached at least the level at which this sample has been taken (250 m). After that time the volcano must have risen considerably. As VERBEEK stated, evidence of rising is still to be found in the terraces along the coast. VERBEEK considers among others G. Batoe as the rest of a coral-reef formed on the volcano-body. It does not seem unreasonable to suppose, that coral-reefs have been formed in the time, when the sea covered big parts of the volcano-body, but whether G. Batoe indeed represents such a rest must be left open to question.

Concerning the distribution of the different rock-types on the island it turns out, that phonolites are much more widely distributed, than indicated on VERBEEK's map. Among others phonolite has been found at G. Nangka, G. Besar, G. Limpang Kopeng and P. Menoeri.

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Geology. — *Ueber Spilite und verwandte Gesteine von Timor.* Von W. P. DE ROEVER. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of April 27, 1940.)

Während der geologischen Expedition der Universität Amsterdam nach den Kleinen Sunda-Inseln im Jahre 1937, unter Leitung von Prof. Dr. H. A. BROUWER, wurden von mir geologische Untersuchungen im Gebiete westlich und südwestlich des Moetisgebirges [1]) ausgeführt, dessen Gipfel der höchste von Niederländisch Timor ist.

In diesem Gebiete wurden Spilite und verwandte Gesteine gefunden, die bisher noch nicht von dieser Insel bekannt waren. Die meisten Eruptivgesteine dieses Gebietes gehören zu einem Komplex, der wahrscheinlich in ungefähr ähnlicher Weise wie die Paleloserie [2]) mit den kristallinen Schiefern verknüpft ist. Die Gesteine dieses Komplexes wurden vielfach als Liegendes der triadischen Fatoekalke gefunden.

Ausser spärlichen Kalken gehören zu diesem Komplex mannigfach gegliederte Eruptivgesteine, nämlich Spilite, Spilitaugitite, Keratophyre, albitisierte Dolerite, Albitite, Basalte, Andesite, Gabbros, Lherzolithe und Serpentine. Vielleicht gehört auch ein Teil der Radiolarite des untersuchten Gebietes zu diesem Komplex. Die Gesteinsgruppe zeigt eine grosse Aehnlichkeit mit den Ophiolithen der Liguriden des Appennins [3]).

Ueber das Alter der Gesteine kann bisher nicht viel gesagt werden; sie sind prätertiären Alters.

Die Gesteine von spilitischer Verwandtschaft zeigen für den Begriff der Spilitfrage lehrreiche Erscheinungen. Erstens ist das Zusammenvorkommen von Basalten und Andesiten mit Spiliten und Keratophyren zu betonen. Weiter sind in den Gesteinen dieses Komplexes vielfach Albitisierungserscheinungen beobachtet worden. Besonders die Dolerite, Andesite und Gabbros zeigen eine deutliche Albitisierung der basischen Plagioklase. Die Feldspäte dieser Gesteine sind meistens zonar gebaut, mit anortitreicheren Kernen. Die basischen Kernfeldspäte werden zuerst von den albitisierenden Agentien angegriffen, wie auch von BAILEY und GRABHAM [4]) beschrieben worden ist. Die Albitisierung der Dolerite ist mitunter mit Analcitisierung

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[1]) Für Ortsnamen wird die niederländische Schreibweise gebraucht, wobei oe = u.

[2]) Für Definition und Beschreibung der tektonischen und stratigraphischen Gesteinsfolgen siehe: D. TAPPENBECK, Geologie des Mollogebirges und einiger benachbarter Gebiete. Geol. Exp. to the Lesser Sunda Islands under leadership of H. A. BROUWER. Amsterdam, **1** (1940).

[3]) G. STEINMANN, Die ophiolithischen Zonen in den mediterranen Kettengebirgen. C.R. XIVe Congr. Géol. Int., **2**, 637—668 (1926).

[4]) E. B. BAILEY und G. W. GRABHAM, Albitization of basic plagioclase felspars. Geol. Mag., Dec. 5, **6**, 250—256 (1909).

verknüpft. In den Gabbros, die sehr basische Plagioklase enthalten, also eukritischen Charakters sind, ist die Albitisierung der Feldspäte ein untergeordnetes Phänomen neben vielen andern Umwandlungserscheinungen. Die Keratophyre, die nur in der Grundmasse Quarz enthalten, sind zweifellos aus Andesiten entstanden, da mehrere Uebergangsgesteine gefunden worden sind. Diese Gesteine enthalten keine Alkaliamphibole oder Alkalipyroxene.

Die Spilite enthalten neben kleinen idiomorphen Albitleisten meistens Pseudomorphosen nach Olivin als Einsprenglinge. Die Grundmasse wird von Augit und Erz gebildet. Skelettförmige, fiederartige und gekrümmte Kristalle von Augit, welche auf rasche Kristallisation hinweisen, sind öfters vorhanden. In den Spiliten wurden keine Reste von ursprünglichen basischen Plagioklasen beobachtet. In den Basalten aber ist örtlich eine anfangende Umwandlung der Feldspäte zu beobachten, welche teilweise sicherlich als Albitisierung zu deuten ist. Die Basalte enthalten ebenfalls Olivinpseudomorphosen.

Ein eigentümlicher Gesteinstypus wird von den Spilitaugititen gebildet. Neben spärlichen, kleinen Albitleisten sind kleine Augitkristalle als Einsprenglinge vorhanden. Die Grundmasse wird fast ausnahmslos von fiederartig entwickeltem oder sphärolithischem Augit zusammengestellt. Wegen des Fehlens basischer Plagioklase sind diese Gesteine nicht genau zu vergleichen mit den „Spilitaugititen" BACKLUNDS[5]) von Nowaja Semlja, welche als Feldspäte sporadisch Andesin-Labradorit mit Albiträndern enthalten. Auch SLAVIK[6]) hat schon ultrabasische spilitische Gesteine von Böhmen erwähnt.

Seit 1909 und 1911, nach der Veröffentlichung der Publikationen von BAILEY und GRABHAM[7]) und von DEWEY und FLETT[8]) haben Albitisierung und spilitische Gesteine in den petrologischen Betrachtungen viel an Bedeutung gewonnen. Von den letztgenannten Autoren wurden diese natronreichen Gesteine als eine Gesteinssippe für sich („spilitic suite") betrachtet, welche den Albit einem postvulkanischen oder juvenilen Prozess verdankte. BENSON[9]) betrachtete den Natronfeldspat der Spilite von New South Wales meistens als primär, welche Meinung später von GILLULY[10]) kritisiert worden ist. Eines der wichtigsten Argumente

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5) H. G. BACKLUND, Die Magmagesteine der Geosynklinale von Nowaja Semlja. Report of the scientific results of the Norwegian expedition to Nowaya Zemlya 1921, N⁰. 45, Oslo (1930).

6) F. SLAVIK, Les "pillow-lavas" algonkiennes de la Bohême. C.R. XIVe Congr. Géol. Int., **4**, 1389—1395 (1926).

7) Loc. cit.

8) H. DEWEY und J. S. FLETT, Some British pillow-lavas and the rocks associated with them. Geol. Mag., Dec. 5, **8**, 202—209 und 241—248 (1911).

9) W. N. BENSON, The geology and petrology of the Great Serpentine Belt of New South Wales, Part IV. Proc. Linnean Soc. N.S.W., **40**, 121—173 (1915).

10) J. GILLULY, Keratophyres of Eastern Oregon and the spilite problem. Amer. Journ. Sc., Ser. 5, **29**, 225—252 und 336—352 (1935).

BENSONs war das Zusammenvorkommen des Albits mit frischem Augit. Wie auch schon von anderen Autoren betont wurde, ist das aber kein entscheidendes Merkmal. In den timoresischen Gesteinen werden teilweise albitisierte Feldspäte von noch ganz frischem Augit begleitet.

Später sind von mehreren Autoren beide Auffassungen über die Entstehung spilitischer Gesteine gestützt und modifiziert worden. Eine der wichtigsten Schwierigkeiten bei der Erklärung dieser Gesteine ist das Vermögen des Albits um molekulare Pseudomorphosen nach den basischen Feldspäten zu bilden, welche so vollständig Struktur und Eigentümlichkeiten der ursprünglichen Feldspäte nachahmen, dass irgendeine Andeutung von sekundärer Entstehung des Albits oft zu fehlen scheint.

Erwähnenswert sind noch die spilitartigen Gesteine von Nowaja Semlja, die von BACKLUND [11]) beschrieben worden sind, welche aber meistens nicht Albit, sondern etwas basischere Plagioklase enthalten.

GILLULY [12]) hat 1935 das Spilitproblem im Zusammenhang mit einer Beschreibung der permischen Keratophyre von Ost Oregon eingehend behandelt. Die Auffassung, dass die spilitischen Gesteine unabhängig von Alkalikalkgesteinen eine eigene Gesteinssippe bilden, ist von diesem Autor kritisiert worden. In vielen Gebieten sind nämlich, wie auch im Moetisgebiete, normale Alkalikalkgesteine zusammen mit Spiliten gefunden worden. Ausserdem zeigen die chemischen Analysen der spilitischen Gesteine nach der Meinung dieses Autors einen vollständigen Uebergang zu denen der normalen verwandten Alkalikalkgesteine. GILLULY hat auch betont, dass die Gesteine spilitischer Verwandtschaft in vielen Gebieten noch Reste basischer Feldspäte enthalten. Auch in den timoresischen Gesteinen ist das der Fall. Nicht nur der erwähnten praktischen Argumente wegen, sondern auch auf theoretischen Gründen ist das primäre Vorkommen von Albit in diesen Gesteinen abzulehnen. GILLULY kommt zu dem Schluss, dass Spilite und Keratophyre als albitisierte Alkalikalkgesteine zu betrachten sind.

Nach meiner Auffassung sind die timoresischen spilitischen Gesteine auch durch Albitisierung aus Alkalikalkgesteinen entstanden.

Ueber Zeit und Ursprung der Albitisierung ist weniger Bestimmtes zu sagen. Im timoresischen Gebiete stehen die albitisierenden Agentien wahrscheinlich in genetischem Zusammenhang mit Albititen. Einige der albititischen Gesteine sind sicher durch Albitisierung von Alkalikalkgesteinen entstanden, aber die meisten Typen dieser Gesteine zeigen keine Andeutung einer sekundären Herkunft des Albits. Die letztgenannten Gesteinstypen gehören zu Augitalbititen und Amphibolalbititen. Der Feldspat ist hier ausnahmslos Albit. Diese albititischen Gesteine sind teilweise mit den schwedischen Pyroxenalbititen oder Värnsingiten zu

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[11]) Loc. cit.
[12]) Loc. cit.

vergleichen, welche von SOBRAL [13]) und VON ECKERMANN [14]) beschrieben worden sind. In einem der Amphibolalbitite sind Alkaliamphibole in untergeordneter Menge vorhanden, während auch Mineralien der Epidotgruppe in wichtiger Menge auftreten. Das Vorkommen der letztgenannten Mineralien ist wahrscheinlich zu vergleichen mit dem der helsinkitischen Gesteine [15]).

Oertlich wurden noch veränderte Albitite gefunden, welche ebenfalls als Feldspat nur Albit enthalten, der kataklastisch beeinflusst worden ist. Als wichtige ursprüngliche Gemengteile sind weiter noch diopsidischer Augit und Aegirinaugit zu erwähnen. Biotit und Alkaliamphibole (z. T. crossitisch) sind teilweise als Neubildungen vorhanden. Eigentümlich ist das Auftreten von Alkaliamphibolen mit kleinem oder sehr kleinem Achsenwinkel. Das Gestein enthält ebenfalls Epidot und Klinozoisit, und ist reich an Titanit und Apatit. Bei der Entstehung der Mineralien der Epidotgruppe haben auch in diesem Gestein wahrscheinlich ähnliche Prozesse eine Rolle gespielt wie diejenige, die bei den Helsinkiten auftreten.

Das Problem, ob auch diese Gesteine albitisierte Alkalikalkgesteine darstellen, ist schwer zu lösen. Einige der Albitite zeigen Strukturen, die schwierig mit einer sekundären Herkunft des Albits in Einklang zu bringen sind, z.B. eigentümliche Verwachsungen von Albit und Augit. Wie aber in den Spiliten zu beobachten ist, können die Albitpseudomorphosen nach basischen Feldspäten so vollkommen sein, dass irgendwelche Andeutung eines Albitisierungsprozesses zu fehlen scheint.

Mehr komplizierte Verhältnisse zeigen die Eruptivgesteine einer anderen Gesteinsfolge, aus dem Perm der Sonnebaitserie. Die sauren Glieder dieser permischen Eruptivgesteine werden im untersuchten Gebiete von Alkalitrachyten gebildet, welche neben Anorthoklas, Albit und Aegirin, als mineralogisches Kuriosum für diese Gesteine einen crossitischen Amphibol enthalten. Die basischen Glieder sind meistens als Melaphyre beschrieben worden. BROUWER [16]) erwähnte schon die kleinen Auslöschungswinkel der Feldspäte in ähnlichen Gesteinen von Timor. Wahrscheinlich gehören die basischen Gesteine des untersuchten Gebietes z.T. auch zu spilitartigen Gesteinen, die aber möglicherweise aus eigentümlichen Orthoklas-Olivin-Laven entstanden sind, welche zu derselben Serie gehören. Einige Uebergangsstadien zwischen Orthoklas-Olivin-Laven und spilitartigen Gesteinen sind auch beobachtet worden. Die Verhältnisse sind aber wegen

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[13]) JOSÉ M. SOBRAL, Contributions to the geology of the Nordingrå Region. Upsala (1913).

[14]) H. VON ECKERMANN, A contribution to the knowledge of late sodic differentiates of basic eruptives. Journ. Geol. **46**, 412—437 (1938).

[15]) Siehe: O. MELLIS, Zur Genesis des Helsinkits. Geol. För. Stockh. Förh. **54**, 419—435 (1932).

[16]) H. A. BROUWER, Gesteenten van Oost Nederlandsch Timor. Jaarb. Mijnwezen Nederl. Oost Indië 1916, Verh. I, 69—260, siehe besonders S. 193—195.

der starken Verwitterung der Gesteine schwierig zu deuten. Vielleicht werden chemische Analysen des Alkaligehaltes der weniger verwitterten Glieder dieser feinkörnigen Gesteine noch nähere Aufklärung über die komplizierten Verhältnisse in dieser Serie geben.

In einem dritten Komplex (Keknenoserie) wurden örtlich viele grosse Eruptivblöcke (bis 3 m in Diameter) gefunden, die vielleicht dem Perm dieser Serie entstammen. Diese Gesteine gehören auch zu den Albititen. Teilweise sind diese Eruptivgesteine sicher als Ganggesteine zu betrachten, da etwas feinkörnigere Gänge in den grossen Eruptivblöcken auftreten. Die Gesteine sind in etwa ähnlicher Weise entwickelt wie die aus dem erstbeschriebenen Komplex. Der Feldspat ist ausnahmslos Albit. Als dunkle Gemengteile spielen Augit und brauner Amphibol eine wichtige Rolle. Die Pyroxenkristalle zeigen öfters unregelmässige Ränder von Aegirin, während die Amphibole mitunter eine sehr schmale Randzone von crossitischem Amphibol erkennen lassen.

Die Albitite mit Alkaliamphibolen und Alkalipyroxenen, welche dieser Serie entstammen, sind sicher nicht von metamorpher Herkunft. Die Gesteine zeigen eine weniger komplizierte Zusammenstellung als die Albitite des erstbeschriebenen Komplexes, z.B. ohne beträchtliche Mengen von Mineralien der Epidotgruppe. Es ist wohl wünschenswert, diese Gesteine, welche auch vom Mother Lode District in Kalifornien [17]) beschrieben worden sind, in Gegensatz zu den Albititen ohne Alkaliamphibole oder Alkalipyroxene als einen Gesteinstypus für sich abzusondern.

Die Gesteinstypen der drei genannten Komplexe sind also alle durch einen Reichtum an Natronfeldspat ausgezeichnet, während nur das Perm der Sonnebaitserie auch viel Kalifeldspat enthält.

Eine eingehendere Beschreibung der verschiedenen Gesteinstypen wird in einer späteren Publikation gegeben werden.

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[17]) Siehe z.B.: A. KNOPF, The Mother Lode System of California. U.S.G.S. Prof. Paper 157 (1929).

Microbiology. — *Manometric investigations on bacterial denitrification.* By E. VAN OLDEN. (Communicated by Prof. A. J. KLUYVER.)

(Communicated at the meeting of April 27, 1940.)

Introduction.

In the course of a study on denitrification with cellulose as a substrate, it was found desirable to carry out some preliminary experiments regarding the mechanism of denitrification with simpler hydrogen donators.

Until now denitrification — which term is used here in the restricted sense of nitrate reduction leading to nitrogen evolution — has nearly always been studied in growing cultures. In order to simplify the problem, it seemed of importance to attempt to study in experiments of short duration the behaviour of a given population of denitrifying bacteria towards nitrate, both in the absence and in the presence of suitable organic substrates. This led to the testing of the applicability of the manometric method for this purpose [1]).

It will be clear that denitrification, *i.e.*, the dissimilation process by which hydrogen is transferred from an organic substrate to nitrate as a hydrogen acceptor, can only be studied with the aid of the manometric method, if nitrogen production is a true measure for the oxidation of the substrate. As soon as accumulation of intermediate non-gaseous products of the nitrate reduction would occur, the manometric readings will not give any more a correct indication of the oxidation of the substrate.

The question whether under the conditions of the experiment denitrification is restricted indeed to the conversion:

$$5\,AH_2 + 2\,HNO_3 \rightarrow 6\,H_2O + N_2 + 5\,A$$

can be decided by studying the course of the gas evolution as a function of time. If nitrogen production starts immediately after adding the nitrate, and if a rectilinear relation between nitrogen production and time exists, it is highly improbable that an accumulation of intermediate stages of the nitrate reduction takes place.

However, if nitrogen production is delayed and the rate of gas evolution increases slowly with time, this indicates that nitrate reduction does not lead directly to free nitrogen, and that accumulation of some intermediate stage — probably nitrite or hyponitrite — occurs.

A second possible complication seemed to be the following.

In manometric experiments on dissimilation the conditions are usually

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[1]) In a recent paper of YAMAGATA (*7b*) the author mentions in a footnote, that he also has successfully applied the Warburg technique to the study of the process in question.

chosen in such a way that they are as unfavourable as possible for growth and proliferation of the cells. For this purpose use is made of bacterial suspensions of great density, whilst as a rule the medium is devoid of any nitrogen source. However, it results from the investigations made by BARKER (2), GIESBERGER (5) and CLIFTON (3), that even under such conditions assimilation is not altogether prevented.

Under the conditions of my experiments, where nitrate had to be added in order to act as a hydrogen acceptor, there seemed to be great danger that a non-negligible part of the nitrate would be involved in assimilatory processes, and in doing so, would seriously interfere with the study of the dissimilation process.

As will appear from the following pages, it proved possible to find conditions under which the discussed complications could be avoided.

Methodical remarks.

All experiments were carried out with *Micrococcus denitrificans* BEIJERINCK. Initially a strain from the collection of the "Laboratorium voor Microbiologie" at Delft was used, which was the same strain as has extensively been studied by ELEMA in connection with his investigations on the redox potentials occurring in media containing denitrifying bacteria. ELEMA (4) has given experimental proof that under certain conditions this strain is able to reduce nitrate quantitatively to nitrogen with simultaneous oxidation of various simple organic hydrogen donators to carbon dioxide and water.

However, this strain proved to have some drawbacks which will be discussed in the following chapter.

For this reason a number of cultures of denitrifying bacteria were tested on their respiratory activity, both in the absence and in the presence of substrates, and on their denitrifying ability in media containing different substrates.

As a result of this investigation another strain of *M. denitrificans*, isolated from an enrichment culture in a cellulose nitrate medium, was chosen for the further experiments.

For details concerning the manometric method of Warburg I refer to: DIXON, Manometric methods, Cambridge 1937.

The bacteria were collected by washing the crop of several agarplate-cultures either with distilled water, or with 0.9 % NaCl solution, and by centrifuging these suspensions in an ordinary centrifuge at the speed of 4000 r.p.m. In some cases they were obtained from a culture in a liquid medium by centrifuging in a Sharples Super-centrifuge at the speed of 35000 r.p.m. The centrifuged bacterial mass was washed twice with water, and either finally suspended in a Na_2HPO_4—KH_2PO_4 buffer solution of pH 7.1 and molarity 1/15, or suspended in distilled water in which case enough concentrated buffer solution was given in the Warburg vessel to yield also a solution of molarity 1/15.

The volume of the liquid in the Warburg vessel was always 3 cc. All manometric experiments were made in duplicate, at 30° C.

In the denitrification experiments the vessels were filled with nitrogen gas which was first led over a glowing copper spiral to free it from traces of oxygen, always present in the commercial product. The nitrate was added from one of the sidebulbs of the vessel at the same time as the substrate, or the substrate was already present in the main vessel in contact with the bacterial suspension, and the nitrate only added, when the vessels had attained temperature equilibrium in the waterbath. The substrate was always given in excess to the nitrate in order to enable the bacteria to reduce all nitrate present to nitrogen.

In the interpretation of the pressure changes observed it was assumed that under anaerobic conditions with nitrate as hydrogen acceptor the gases evolved were only carbon dioxide and nitrogen. As will be seen, the results obtained justify this assumption. The separate estimation of the gases in question was accomplished in the usual way by determining both the gas evolution in the presence of 0.2 cc 20 % potassium hydroxyde in one manometer vessel, and in another vessel in the absence of KOH.

Qualitative reactions on nitrite were performed with the aid of GRIESS–ROMIJN's nitrite reagent. The presence of nitrate was tested after TILLMANS, the nitrite, if present, being destroyed by boiling the solution with urea and sulfuric acid.

Regarding the presentation of the results in Tables 1 to 5 it needs scarcely be remarked that the term Q_{O_2} is the amount of oxygen consumed in one hour by a quantity of bacteria corresponding to one mg. dry weight.

An analogous term can be introduced to describe the rate of denitrification: Q_{N_2}. Q_{N_2} is the amount of nitrogen liberated by the reduction of nitrate under anaerobic conditions in one hour by a quantity of bacteria corresponding to one mg. dry weight.

For the sake of comparison between aerobic oxidation and oxidation by means of nitrate oxygen, it must be observed that the amount of gaseous nitrogen produced, multiplied by 2.5, is equal to the nitrate oxygen consumed in the oxidation of the substrate, assuming that the equation: $2\,HNO_3 = H_2O + N_2 + 5\,O$ holds.

Experimental.

As already mentioned the first experiments were made with the strain of *Micrococcus denitrificans* from the collection of the "Laboratorium voor Microbiologie" at Delft.

The bacteria used in the first experiment were cultivated aerobically on tapwater-glycerol-agar containing 0.1 % nitrate. In the manometric experiments gas evolution only started two hours after the nitrate had been added, both in the absence and in the presence of the glycerol.

In the vessel in which both glycerol and nitrate were present, the rate of gas production slowly increased till after 5½ hours the rate rapidly

decreased. It appeared that at the end of the experiment the theoretical amount of nitrogen had been evolved; in agreement herewith neither nitrate nor nitrite were found.

When only nitrate — but no substrate — had been added to the bacterial suspension the rate of nitrogen evolution was almost constant, very slowly decreasing at the end of the experiment. The nitrogen produced was only part of the theoretical amount which could be expected from the nitrate added. At the end both nitrate and nitrite were present.

The second experiment was made with bacteria cultivated in a liquid medium containing 2 % glycerol and 0.5 % potassium nitrate, in which medium active denitrification took place during the growth of the bacteria.

This time nitrogen production started immediately after the addition of the nitrate. Moreover the rate of the gas evolution was constant, both in the absence and in the presence of glycerol. When glycerol was present, the rate of the gas production was somewhat more rapid than in the presence of nitrate only. In both cases the rate of the nitrogen production only decreased at the moment that almost the theoretical amount of gas was evolved. In the absence of nitrate no gaseous metabolism was observed under anaerobic conditions.

In a way these results may be deemed quite satisfactory, in so far as they prove that it is possible to find conditions under which nitrogen evolution — manometrically established — is a correct measure of the rate of denitrification. Nevertheless, the fact that there was only a small difference in the rate of nitrogen evolution in the experiments with and without substrate was quite unexpected, and undesirable from the standpoint of getting a clear insight into the nature of the dissimilation process. The only possible explanation of this phenomenon seemed to be that the bacteria in question were able to oxidize their own reserve materials with the aid of the nitrate added, and that we are dealing here with a process which may be termed: "endogenous denitrification". Until now this process has escaped the attention of the investigators who have confined themselves almost completely to a study of denitrification in growing cultures.

The correctness of the interpretation given is strongly supported by the observation that the bacteria were also characterized by a high endogenous respiration.

In order to avoid the difficulty of the high endogenous denitrification steps were taken to find, if possible, a strain which did not show this complication, and as a result hereof the further experiments were made with a strain of *Micrococcus denitrificans*, already referred to in the preceding chapter.

The result of the first experiment as recorded in Table 1 was not encouraging. No gas production whatever was observed when a suspension of bacteria grown on plain peptone agar was brought together with nitrate under anaerobic conditions. Even no nitrite, the occurrence of which is the first indication of nitrate reduction, could be detected. Aerobically the

TABLE 1.
Micrococcus denitrificans.

Denitrification and respiration.

Substrate: Na-acetate 0.2 cc of a 5 % solution. KNO_3 added in the anaerobic experiments: 0.4 cc of a 0.5 % solution (corresponding with a nitrogen production of 221 mm^3, 0° C. and 760 mm Hg).

Bacteria cultivated aerobically on peptone agar without nitrate; 48 hours at 30° C.

	In air				In nitrogen, with nitrate			
	with substrate		without substrate		with substrate		without substrate	
	oxygen consumed mm^3	Q_{O_2}	oxygen consumed mm^3	Q_{O_2}	nitrogen produced mm^3	Q_{N_2}	nitrogen produced mm^3	Q_{N_2}
0— 60 min.	373	47	43	5	6	0.4	8	0.3
60—120 „	496	63	30		2		—1	
120—180 „	525	67	30		—2		0	
180—240 „	173	22	22	2.8	4		3	
240—300 „	(118)	(15)	(16)		4		2	
Total	1685		141		14 [1])		12 [1])	

[1]) Nitrate present, no nitrite formed.

bacteria used behaved normally and oxidized the substrate: sodium acetate rapidly.

Hereupon, the same experiment was repeated with bacteria, grown aerobically on peptone agar containing 0.5 % potassium nitrate. (Table 2). However, only a slow nitrogen production was observed, when nitrate was added to the suspension already in contact with the substrate. Moreover, the rate of this gas evolution showed a distinct rise with time, whilst the final volume of the nitrogen obtained corresponded only to a part of the nitrate added.

Although these results meant real progress, when compared with the outcome of the first experiment, they still were unsatisfactory, because of the inconstant rate of gas evolution and because of the incomplete conversion of the nitrate added. That this behaviour was not due to a general lack of activity of the bacteria was clearly shown by the fact that the aerobic oxidation of sodium acetate proceeded at a quite normal rate, the Q_{O_2} being no less than 60.

The fact that the bacteria which had been cultivated in the absence of nitrate (Experiment 1) were quite unable to denitrify in the "resting state", whilst the bacteria which had been cultivated in the presence of nitrate (Experiment 2) showed an unmistakable activity in this respect,

TABLE 2.

Micrococcus denitrificans.

Denitrification and respiration.

Substrate: Na-acetate 0.2 cc of a 5 % solution. KNO_3 added in the anaerobic experiments: 0.4 cc of a 0.5 % solution (corresponding with a nitrogen production of 221 mm³, 0° C. and 760 mm Hg).

Bacteria cultivated aerobically on peptone agar, containing 0.5 % KNO_3; 48 hours at 30° C.

	In air				In nitrogen, with nitrate		
	with substrate		without substrate		with substrate		without substrate
	oxygen consumed mm³	Q_{O_2}	oxygen consumed mm³	Q_{O_2}	nitrogen produced mm³	Q_{N_2}	nitrogen produced mm³
0— 60 min.	340	44	37		12	1.5	4
60—120 „	457	59	34		9	1.2	—1
120—180 „	467	60	20		14	1.8	0
180—240 „	281	36	19		24	3.0	5
240—300 „	(130)	(17)	(30)		22	2.8	0
Total	1675		140		81 [1]		8 [1]

[1]) Nitrate present, nitrite formed.

TABLE 3.

Micrococcus denitrificans.

Denitrification.

Substrate: Na-acetate 0.2 cc of a 5 % solution. KNO_3 added: 0.4 cc of a 0.84 % solution (corresponding with a nitrogen production of 372 mm³, 0° C. and 760 mm Hg).

Bacteria cultivated on peptone agar containing 2 % KNO_3; 48 hours in an anaerobic jar at 30° C.

	In nitrogen, with nitrate			
	with substrate		without substrate	
	nitrogen produced mm³	Q_{N_2}	nitrogen produced mm³	Q_{N_2}
0— 30 min.	112	33.4	45	13.4
30— 60 „	78	23.3	20	5.97
60— 90 „	75	22.4	20	5.97
90—120 „	62	18.5	20	5.97
120—180 „	17	2.5	37	5.5
180—240 „	0	0	(26)	
Total	344 [1]		168 [2]	

[1]) Nitrate and nitrite absent.

[2]) Nitrate present, nitrite trace.

made it seem probable that a still better activity could be expected for bacteria which had been completely depending on denitrification during the period of their growth.

For this reason a third experiment was made with bacteria, cultivated anaerobically on peptone agar containing 2 % nitrate. Table 3 shows that these bacteria were indeed capable of a very vigorous denitrification. After a short initial period of abnormal high activity the rate of nitrogen production arrived at a fairly constant level which was maintained till almost the theoretical amount of gas was produced. Then the rate rapidly declined to zero. In this case a Q_{N_2} was observed which corresponded with a Q_{O_2} of the same order of magnitude as that actually observed in the previous aerobic experiments.

The results of the experiments given in Tables 1, 2 and 3 are represented graphically in Figure 1. In this figure the difference in bacterial quantities added pro vessel is accounted for.

The figure clearly demonstrates the marked difference in denitrifying ability of the bacteria in its dependence on cultural conditions.

It seemed desirable to corroborate these results in some further experi-

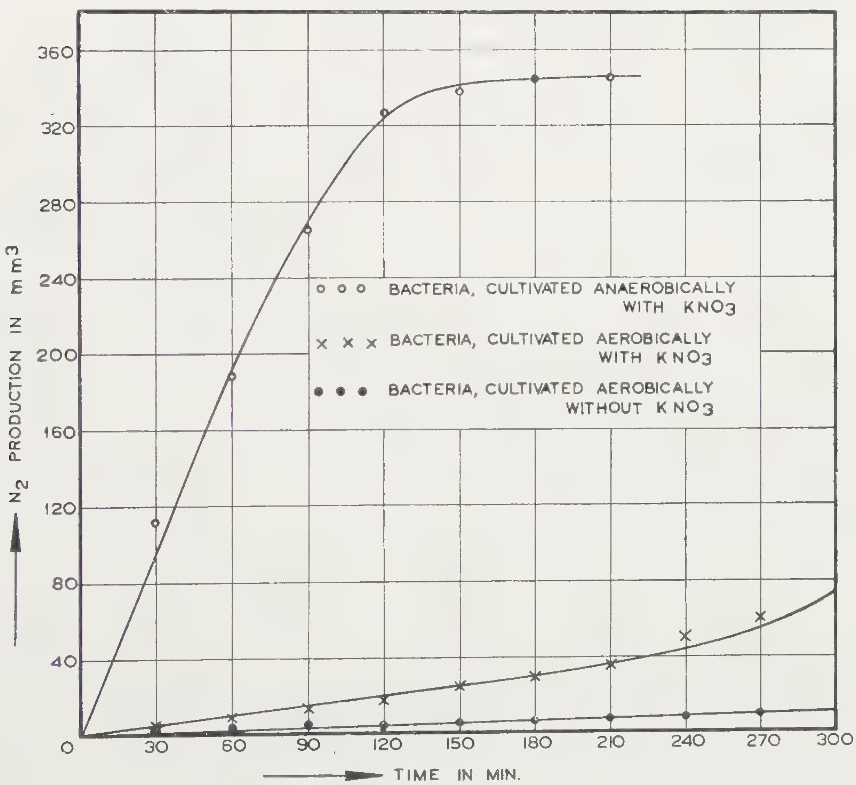

Fig. 1.

Micrococcus denitrificans; denitrification with Na-acetate as substrate.

ments. For this purpose comparative experiments were made with bacteria partly grown aerobically on peptone agar with 2 % nitrate, partly on the same medium in an atmosphere of pure hydrogen. As it appears from Tables 4 and 5 it was again found that only the latter bacteria were able to bring about denitrification at a rate of the same order of magnitude as that of the aerobic dissimilation ($Q_{O_2} = 2.5\ Q_{N_2}$!).

The bacteria cultivated aerobically only slowly produced nitrogen from the nitrate added, the rate of the gas evolution, moreover, increasing with time. At the end of the experiment nitrate was left, and nitrite was shown to be present in the medium.

In contrast herewith, the bacteria cultivated anaerobically immediately started to reduce the nitrate to free nitrogen in the experiment with substrate. During the first two hours the rate of nitrogen evolution was practically constant, and then quickly dropped to zero. The nitrogen produced corresponded with the theoretical amount for complete conversion of the nitrate. Neither nitrate, nor nitrite, were present in the medium at the end of the experiment.

TABLE 4.

Micrococcus denitrificans.

Denitrification and respiration.

Substrate: Na-acetate 0.2 cc of a 5 % solution. KNO_3 added in the anaerobic experiments: 0.4 cc of a 0.84 % solution (corresponding with a nitrogen production of 373 mm³, 0° C. and 760 mm Hg).

Bacteria cultivated aerobically on peptone agar, containing 2 % KNO_3; 48 hours at 30° C.

	In air				In nitrogen, with nitrate			
	with substrate		without substrate		with substrate		without substrate	
	oxygen consumed mm³	Q_{O_2}	oxygen consumed mm³	Q_{O_2}	nitrogen produced mm³	Q_{N_2}	nitrogen produced mm³	Q_{N_2}
0— 30 min.	209	46.1	37	8.2	6	0.66	9	0.66
30— 60 „	253	55.8	34		0		—3	
60— 90 „	305	67.2	22		6	2.0	6	
90—120 „	388	85.5	18		12		9	
120—180 „	427	47.1	39		14	1.54	0	
180—240 „	173	19.1	37		19	2.09	5	
240—300 „	166	18.3	29	3.2	27	2.97	3	
Total	1921		216		84[1])		29[1])	

[1]) Nitrate present, nitrite formed.

TABLE 5.

Micrococcus denitrificans.

Denitrification and respiration.

Substrate: Na-acetate 0.2 cc of a 5 % solution. KNO_3 added in the anaerobic experiments: 0.4 cc of a 0.84 % solution (corresponding with a nitrogen production of 373 mm³, 0° C. and 760 mm Hg).

Bacteria cultivated anaerobically on peptone agar containing 2 % KNO_3; 48 hours at 30° C.

	In air				In nitrogen, with nitrate			
	with substrate		without substrate		with substrate		without substrate	
	oxygen consumed mm³	Q_{O_2}	oxygen consumed mm³	Q_{O_2}	nitrogen produced mm³	Q_{N_2}	nitrogen produced mm³	Q_{N_2}
0— 30 min.	171	60.8	41	14.5	96	34.1	29	10.3
30— 60 „	204	70.7	17		79	28.1	12	4.3
60— 90 „	255	90.6	20		82	29.1	8	2.8
90—120 „	266	94.5	6		88	31.3	14	5.0
120—180 „	502	89.2	11	1.9	26	4.6	25	4.4
180—240 „	233	41.4	+2		0		22	3.9
240—300 „	120	21.3	+2		0		10	1.8
Total	1751		91		371 [1])		120 [2])	

[1]) Nitrate and nitrite absent.

[2]) Nitrate present, nitrite absent.

In the experiment in which only nitrate had been added to the suspension, but no substrate, the decreasing rate of the nitrogen production showed the rapid disappearing of the reserve materials present in the cells. It is worth-noticing that nitrite was absent at the end of this experiment.

Discussion of results.

As has been remarked in the Introduction the aim of the investigation made was to test the possibility of studying denitrification of "resting bacteria" with the aid of the manometric method.

The foregoing experiments show that, indeed, under certain conditions denitrification with "resting cells" of *Micrococcus denitrificans* can be obtained which answers the requirements made. Hereto it was found necessary to use cells which during their development had depended on denitrification as a sole source of energy, *i.e.*, which had been grown anaerobically in the presence of nitrate. With these cells in the manometric experiments an evolution of nitrogen could be observed of the same order of magnitude as that of the aerobic gaseous metabolism. Moreover, the

rate of nitrogen production proved to be constant over a long period, namely as long as a sufficient amount of nitrate still was present. This constant rate offers sufficient proof that under the condition of the experiment no intermediate stages of the nitrate reduction were formed in any appreciable amount. In agreement herewith the total amount of nitrogen evolved corresponded with the amount which should be obtained from the nitrate added on the basis of a complete conversion.

The constant rate of nitrogen evolution also shows that the anxiety expressed in the Introduction, *viz.*, that assimilatory processes might interfere with the study of denitrification as such, has been unjustified.

From all this we may conclude that in principle the manometric method is suitable for the study of denitrification. Experiments in which besides the nitrogen also the carbon dioxide produced from the substrate has been estimated are already in progress, and the results will be reported in a forthcoming communication.

Two more points should be brought to the front.

In the first place it has been found that with certain bacteria denitrification proceeds almost equally well in the presence and in the absence of an organic substrate. This can only mean that these bacteria are able to oxidize reserve materials present in the cells at the expense of the nitrate added. This process has been termed "endogenous denitrification" in analogy to the well-known process of "endogenous respiration". To a smaller extent this endogenous denitrification is encountered in all cases.

Secondly the investigation has clearly shown the decisive influence of the previous history of the cells on their denitrifying activity, only those cells being appreciably active which originate from cultures in which denitrification has been the main source of energy. In this connection it may be remarked that both AUBEL and GLASER (1) and YAMAGATA (7*a* and *b*) have recently given arguments in favour of the view that denitrification is depending on the presence of a special nitrate activating enzyme system in the cells. The experiments made show, therefore, that this "nitrate reductase" must be considered as an "adaptive enzyme" in the sense of KARSTRÖM (6).

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(Communicated at the meeting of April 27, 1940.)

15. *System of four spheres lying in the corners of a square.* — The system of 4 spheres can be treated along similar lines as the cubical system; account, however, must be taken of the circumstance that the components of the angular velocity which the system assumes under the influence of a shearing motion of the liquid will be different from those which were found for the cube.

The value of λ for the system of four spheres lying in the corners of a square becomes:

$$\lambda = \frac{1}{4}\left(1 + 2{,}70\,\frac{R}{a} - 0{,}04\,\frac{R^2}{a^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (51)$$

In order to investigate the behaviour of the system and its influence upon the effective viscosity of a liquid in shearing motion, we take the plane of the square as the ξ_1, ξ_2-plane. The ξ_3-coordinates of the centres of all four spheres then will be zero. This makes it possible to define the rotation of the system and the quantities describing the remaining relative motion of the liquid in the following way, which at once refers to the case of a field of flow of general type, as given by eq. (45): — With the aid of eqs. (30) we obtain:

$$\dot{\xi}_i = \sum_{jkl} a_{ik}\, a_{jl}\, \varkappa_{kl}\, \xi_j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

where the summation with respect to j now can be restricted to the terms $j=1$, $j=2$. We consequently can define the components ω_{31} (= — angular velocity about the ξ_2-axis) and ω_{32} (= + angular velocity about the ξ_1-axis) of the square in such a way that the value of $\dot{\xi}_3$ is wholly determined by them. When the components of the angular velocity about the three axes are denoted by p, q, r respectively, we then have:

$$p = \omega_{32} = \sum_{kl} a_{3k}\, a_{2l}\, \varkappa_{kl}; \qquad q = -\,\omega_{31} = -\sum_{kl} a_{3k}\, a_{1l}\, \varkappa_{kl};$$

and:

$$\dot{\xi}_3 = \omega_{31}\,\xi_1 + \omega_{32}\,\xi_2 = -q\,\xi_1 + p\,\xi_2.$$

*) Continued from these Proceedings **43**, 435 (1940).

Having found the angular velocities p and q, we further are concerned with the motion in the ξ_1, ξ_2-plane only, which motion now can be fully described by means of the following quantities:

$$\left.\begin{array}{l} D_{11} = \sum\limits_{kl} \alpha_{1k}\, \alpha_{1l}\, \varkappa_{kl}; \qquad D_{22} = \sum\limits_{kl} \alpha_{2k}\, \alpha_{2l}\, \varkappa_{kl} \\ D_{12} = \frac{1}{2} \sum\limits_{kl} (\alpha_{1k}\, \alpha_{2l} + \alpha_{2k}\, \alpha_{1l})\, \varkappa_{kl} \end{array}\right\} \quad . \quad . \quad . \quad . \quad (53a)$$

and:

$$\omega_{12} = -r = \tfrac{1}{2} \sum_{kl} (\alpha_{1k}\, \alpha_{2l} - \alpha_{2k}\, \alpha_{1l})\, \varkappa_{kl} \quad . \quad . \quad . \quad . \quad (53b)$$

while D_{13}, D_{23}, D_{33} are all replaced by zero (it thus is no longer possible to make use of the relation $D_{11} + D_{22} + D_{33} = 0$).

Systems of forces f and g then must be found of types related to those indicated in figs. 4 and 5. The calculation of the contribution to η_{sp} either can be based upon that of the strengths of the various dipoles formed by these forces, or upon the determination of the dissipation. When the latter course is chosen, it is found that the expression (44) now reduces to:

$$6\, \pi \eta\, Ra^2\, [k_1\, (D_{11}{}^2 + D_{22}{}^2) + 2\, k_2\, D_{11}\, D_{22} + 2\, l_1\, D_{12}{}^2] \quad . \quad . \quad (54)$$

with values of k_1, k_2, l_1 which of course will be different from those given in (35) and (37), although also in the present case the terms of the highest order of magnitude will be 1, 0, 1 respectively.

The mean value for all positions in space of the square is found to be:

$$6\, \pi \eta\, Ra^2\, I\, (\tfrac{2}{15}\, k_1 - \tfrac{1}{15}\, k_2 + \tfrac{1}{10}\, l_1) \quad . \quad . \quad . \quad . \quad . \quad (55)$$

with I given by (48), from which it can be deduced that the value of Λ_{II} (including the EINSTEIN term) becomes:

$$\Lambda_{II} = \frac{9}{8}\, \frac{a^2}{R^2} \left(\frac{2}{15}\, k_1 - \frac{1}{15}\, k_2 + \frac{1}{10}\, l_1 \right) + 2{,}5 = \frac{21}{80}\, \frac{a^2}{R^2} + 0{,}247\, \frac{a}{R} + 2{,}82. \quad (56)$$

It may be added that in the case of the system of two spheres, considered in section **8**, the expression corresponding to (44) and (54) is:

$$3\, \pi \eta\, Ra^2\, k_1\, D_{33}{}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

with $k_1 = (1 - 3R/2a)^{-1}$ according to eq. (20), while:

$$D_{33} = \sum_{kl} \alpha_{3k}\, \alpha_{3l}\, \varkappa_{kl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

provided the axis of the system is taken along the ξ_3-axis. The average value for all directions in space becomes:

$$3\, \pi \eta\, Ra^2\, I \,.\, \tfrac{1}{15}\, k_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

which leads back to (21).

16. The main term of the expression for A_{II}, i.e. the term of the order a^2/R^2, retains its value also when the mutual influences of the spheres upon each other are neglected and STOKES' formula for the resistance is applied to each sphere as if it were alone. This term therefore can be given in a more general form, valid for systems of spheres of arbitrary type. The method to be followed is that given by HUGGINS for the case of plane motion [32]); it will be generalized here for three-dimensional motion.

It is assumed again that with respect to the system $Ox_1\,x_2\,x_3$ the motion of the liquid is described by eqs. (45). With reference to the system ξ_1, ξ_2, ξ_3, rigidly connected with the system of spheres, the motion then is given by eqs. (52). The ξ_1, ξ_2, ξ_3-system will pick up a rotation, with provisionally unknown components of angular velocity, which again can be described by a set of quantities ω_{ij}, where $\omega_{ij} = -\omega_{ji}$. The components of the relative velocity of the liquid, with respect to the sphere numbered n, then will be given by:

$$(\dot{\xi}_{in})_{rel} = \sum_{jkl} (\alpha_{ik}\, \alpha_{jl}\, \varkappa_{kl} - \omega_{ij})\, \xi_{jn}. \quad . \quad . \quad . \quad . \quad . \quad (60)$$

When now STOKES' formula is applied, the force acting on this sphere will have the components:

$$F_{in} = 6\,\pi\,\eta\, R_n\, (\dot{\xi}_{in})_{rel} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

where R_n is the radius of this sphere. As mentioned before the system of forces must have no resultant, neither a resulting moment. The first condition is satisfied provided:

$$\sum_n R_n\, \xi_{in} = 0 \qquad (i = 1, 2, 3) \quad . \quad . \quad . \quad (62)$$

where the summation with respect to n is extended over all the spheres of the system. The second condition requires that:

$$\sum_n (F_{2n}\, \xi_{3n} - F_{3n}\, \xi_{2n}) = 0, \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

It will be shown that this condition makes it possible to find the magnitudes of the ω_{ij}.

When the system of spheres is replaced by a system of points, having the coordinates ξ_{in} and bearing masses equal to R_n, it is known that the system can always be orientated with respect to the coordinate system ξ_1, ξ_2, ξ_3 in such a way that not only equations (62) will be satisfied, but also the following equations:

$$\sum_n R_n\, \xi_{1n}\, \xi_{2n} = \sum_n R_n\, \xi_{1n}\, \xi_{3n} = \sum_n R_n\, \xi_{2n}\, \xi_{3n} = 0. \quad . \quad . \quad . \quad (64)$$

We further write:

$$\sum_n R_n\, \xi_{in}^2 = A_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

In this case the ξ_1, ξ_2, ξ_3-axes are the principal axes of inertia of the system passing through its centre of gravity, while A_1, A_2, A_3 are its principal moments of inertia.

Equations (63) now can be worked out; omitting the factor $6\,\pi\eta$ it is found that they take the form:

$$\omega_{ij}\,(A_i + A_j) = \sum_{kl} (A_j\, \alpha_{ik}\, \alpha_{jl} - A_i\, \alpha_{jk}\, \alpha_{il})\, \varkappa_{kl} \quad . \quad . \quad . \quad (66)$$

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[32]) M. L. HUGGINS, Journ. of Phys. Chem. **42**, 911 (1938); **43**, 439 (1939). HUGGINS gives attention both to the case of very large and to that of small Brownian motion.

The values of the ω_{ij}, derived from these equations, are substituted into the expressions (60), which then assume the form:

$$(\dot{\xi}_{in})_{rel} = \sum_{jkl} \frac{A_i}{A_i + A_j} (\alpha_{ik}\,\alpha_{jl} + \alpha_{jk}\,\alpha_{il})\,\varkappa_{kl}\,\xi_{jn} = \sum_j \frac{2\,A_i}{A_i + A_j} D_{ij}\,\xi_{jn} \quad . \qquad (67)$$

where the D_{ij} are the quantities already defined by means of eqs. (46) and (45*a*).
The total dissipation is given by:

$$\sum_{in} F_{in}\,(\dot{\xi}_{in})_{rel} = 6\,\pi\,\eta \sum_n R_n\,[(\dot{\xi}_{1n})_{rel}{}^2 + (\dot{\xi}_{2n})_{rel}{}^2 + (\dot{\xi}_{3n})_{rel}{}^2] \quad . \quad . \qquad (68)$$

After substitution of the expressions for $(\dot{\xi}_{in})_{rel}$, this formula is transformed into:

$$\left.\begin{aligned} \textit{dissipation} &= 6\,\pi\,\eta \sum_{ij} \frac{4\,A_i^2\,A_j}{(A_i + A_j)^2}\,D_{ij}{}^2 = \\ &= 6\,\pi\,\eta \left[(A_1\,D_{11}{}^2 + A_2\,D_{22}{}^2 + A_3\,D_{33}{}^2) + \right. \\ &\quad \left. + 4 \left(\frac{A_1\,A_2}{A_1 + A_2} D_{12}{}^2 + \frac{A_1\,A_3}{A_1 + A_3} D_{13}{}^2 + \frac{A_2\,A_3}{A_2 + A_3} D_{23}{}^2 \right) \right] \end{aligned}\right\} . \qquad (69)$$

where use has been made of (64) and (65).
It remains to find the mean value of this expression for all positions in space of the ξ_1, ξ_2, ξ_3-system (considered as equally probable). As we have:

$$\overline{D_{11}{}^2} = \frac{1}{15} I, \qquad \overline{D_{12}{}^2} = \frac{1}{20} I, \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (70)$$

where I has been defined in (48), the following result is obtained:

mean dissipation $=$

$$6\,\pi\,\eta\,I \left[\frac{A_1 + A_2 + A_3}{15} + \frac{1}{5} \left(\frac{A_1\,A_2}{A_1 + A_2} + \frac{A_1\,A_3}{A_1 + A_3} + \frac{A_2\,A_3}{A_2 + A_3} \right) \right] \quad . \quad . \qquad (71)$$

For the system of 8 equal spheres in the corners of a cube we have:

$$A_1 = A_2 = A_3 = 2\,R\,a^2;$$

for the system of 4 spheres in the corners of a square:

$$A_1 = A_2 = R\,a^2; \quad A_3 = 0;$$

for the system of 2 spheres:

$$A_1 = A_2 = 0; \quad A_3 = \tfrac{1}{2}\,Ra^2;$$

for the system of 3 spheres in one line (fig. 1*b*):

$$A_1 = A_2 = 0; \quad A_3 = \tfrac{1}{2}\,Ra^2;$$

for the system of 4 spheres in one line (fig. 1*c*):

$$A_1 = A_2 = 0; \quad A_3 = \tfrac{5}{9}\,Ra^2.$$

It is not difficult to verify that when these results are inserted into (71) we obtain the principal terms (*i.e.* the terms which remain of importance when $R/a \to 0$) of formulae (49), (55), and of the dissipation formulae corresponding to (22), (24), (26) respectively.

The development of a similar general treatment for the case where the mutual influence of the spheres is not neglected might be of interest, but will present a far more difficult problem.

17. The application of the formulae derived to the cases of serum globulin, thyroglobulin and *Octopus* haemocyanin leads to the following results:

serum globulin ($M = 167000$; $V = 0{,}745$).

	$10^8 R$	$10^8 a$	λ	$10^{13} S_{calc}$
2 spheres (fig. 1*a*)	29,1	246	0,559	7,15
3 spheres (fig. 1*b*)	25,4	269	0,438	6,4
4 spheres (fig. 1*c*)	23,1	283	0,381	6,1
4 spheres in a square	23,1	102	0,403	6,5
8 spheres (cube)	18,4	56	0,354	7,3

$10^{13} S_{obs} = 7{,}1$.

thyroglobulin ($M = 650000$; $V = 0{,}72$).

	$10^8 R$	$10^8 a$	λ	$10^{13} S_{calc}$
2 spheres (fig. 1*a*)	45,2	410	0,555	19,5
3 spheres (fig. 1*b*)	39,5	447	0,431	17,4
4 spheres (fig. 1*c*)	35,9	468	0,373	16,5
4 spheres in a square	35,9	169	0,392	17,4
8 spheres (cube)	28,5	94	0,337	18,9

$10^{13} S_{obs} = 19{,}2$.

Octopus haemocyanin ($M = 2800000$; $V = 0{,}740$).

	$10^8 R$	$10^8 a$	λ	$10^{13} S_{calc}$
2 spheres (fig. 1*a*)	74,2	630	0,559	47,8
3 spheres (fig. 1*b*)	64,7	686	0,437	42,8
4 spheres (fig. 1*c*)	58,9	723	0,381	41,0
4 spheres in a square	58,9	260	0,403	43,4
8 spheres (cube)	46,7	143	0,353	47,9

$10^{13} S_{obs} = 49{,}3$.

For the case of *Octopus* haemocyanin it has been attempted to give an approximate representation to scale of three of the systems in fig. 6. It must be observed that in the case of the cubical system the ratio of R to a has already become so large that the degree of approximation of the calculations may be impaired by it. — It may be noted that when the velocity of the liquid is calculated at the centre of the cube, in the case of a rectilinear motion of the system with a velocity U, as *e.g.* in sedimentation, a value of approximately $0{,}96$—$0{,}99\,U$ is found. This means that

the liquid enclosed between the 8 spheres is carried along almost with the same velocity as the system itself, and it may be inferred that the sedimentation velocity will not be greatly changed, when the space between the spheres should be filled up [33]).

In the results given above the radii of the spheres have been calculated from the volumes of the unhydrated molecules. In consequence of what has been said above it is to be expected that they will not change very

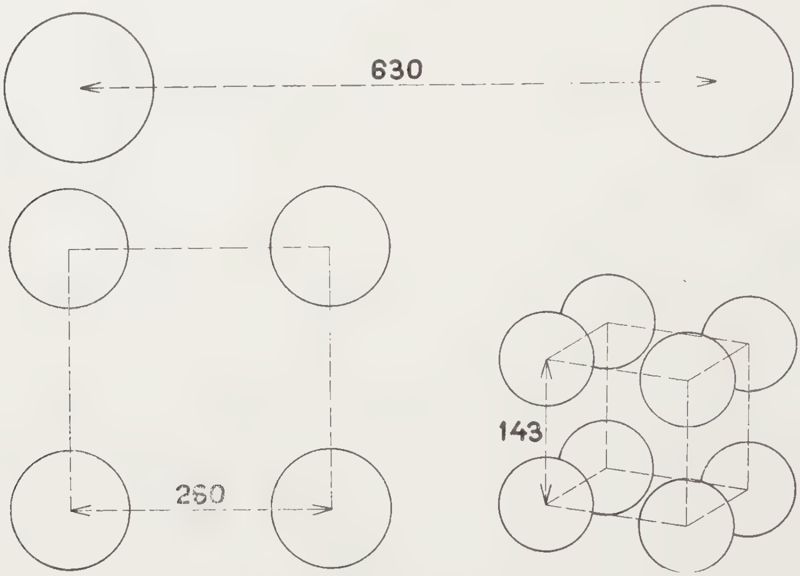

Fig. 6. Systems consisting of 2, 4 and 8 spheres respectively, adapted to the data for *Octopus* haemocyanin. Dimensions in 10^{-8} cm.

much if the hydration water is stored mainly in the space between the spheres.

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[33]) It is of interest to consider the value of the ratio f/f_0 for this case, where f_0 is the frictional constant calculated on the supposition that the molecule has the form of a single sphere (see "The Ultracentrifuge", p. 10). According to form. (13), *loc. cit.*,

$$f_0 = 6\pi\eta N_A R_0,$$

where:

$$R_0 = (3MV/4\pi N_A)^{1/3}.$$

Evidently: $R_0 = 2R$, when R represents the radius of a sphere in the cubical system. Hence, comparing with eq. (15) above, we have:

$$f/f_0 = R/\lambda R_0 = 1/2\lambda,$$

where λ is given by (28). For the proteins considered in the text this becomes: 1,42 — 1,48.

It will be evident that a value of f/f_0 greater than unity, although being an indication of a deviation of the molecule from the true spherical form, not always can be taken as an indication that the form of the molecule should be markedly elongated or flattened.

From the numbers given it follows that the cubical system with 8 spheres — within the limits of the approximation — fits the observed sedimentation velocities practically equally well as the system of two spheres, whereas the others come out less satisfactorily, although all systems fit better than did the ellipsoids.

18. *Conclusions.* — The main part of the present paper has been devoted towards supplementing the formulae collected in Ch. III of the "Second Report on Viscosity and Plasticity" for the specific increase of the viscosity of a suspension of elongated particles, by developing expressions for the cases where the suspended particles

(*a*) have the form of oblate ellipsoids of revolution (section **5** above);
or (*b*) can be represented by systems consisting of a few rigidly connected spheres (sections **7—16**).

In all cases it has been assumed that the Brownian movement is sufficiently intense in order to make all positions in space of the particles equally probable, so that the coefficient calculated always is the one denoted by A_{II}.

Expressions for the calculation of the mean sedimentation velocity have been added in each case.

The results obtained, together with those already given in the "Second Report", have been applied to the experimental data for protein molecules, as found by SVEDBERG and his co-workers. It has been assumed that the values of η_{sp} as measured by POLSON, and the data given for the molecular weights and for the sedimentation constants of the proteins can be regarded as trustworthy; and that they are not materially affected by electroviscous effects, by effects of concentration or by non-linear effects connected with the velocity gradient in viscosity measurements, so that the possibility of a preferential orientation, and that of a deformation of the molecules under the action of the forces due to the shearing motion, are ruled out. It then has been attempted to check the theoretical relation between these quantities, which will follow when a certain shape is assumed for the molecule. For this purpose the values of M, V and η_{sp} have been used to obtain the dimensions of the molecule; from these the sedimentation constant was calculated and compared with the experimental value.

It has been found that the supposition of an elongated rotational ellipsoidal shape of the protein molecule, although leading to the correct order of magnitude for the sedimentation constant, does not fit the experimental results sufficiently closely; discrepancies remain, which appear to exceed the experimental errors.

These discrepancies cannot be removed by assuming that the molecules are hydrated, so long as the assumption of an elongated ellipsoidal form is retained. Nor can they be removed by supposing that the molecules have the form of oblate ellipsoids of revolution.

Comparison with some systems consisting of a few rigidly connected spheres suggests as a probable means for removing the discrepancy to suppose that the mass of the molecule, instead of being concentrated mainly at the centre, is more displaced outwards. It has been found that both a model system consisting of two spheres, and one consisting of 8 spheres at the corners of a cube, may fit the experimental results almost equally well. The calculations themselves do not give the means for deciding between these two cases, but as the system of two spheres does not represent a very probable form, and as the fit becomes less good when 3 or 4 spheres in one line are taken, the cubical system might be preferred. A prismatic type of system, however, perhaps may serve as well. When the molecule should be hydrated, the hydration water in systems of this kind could be stored in such a manner that the exterior dimensions and the sedimentation constant probably will not be greatly affected by it.

The calculations, however, are approximations, and moreover for a deeper study it would be desirable to have data for a greater variety of forms, *e.g.* for prismatic arrangements, for a solid cube and for a solid prism, and for systems of disks, mounted perpendicularly to a common axis. The treatment of such cases will form a set of difficult problems. As the numerical differences between the results to be expected for such systems probably will not be very great, it will be evident that at present there is not much chance to obtain satisfactorily definite estimates of the length/diameter ratio of the molecules.

Finally it appears that POLSON's empirical formula for η_{sp}/cV (compare footnote 9) above), although it might point in the right direction for cylindrical particles, cannot be considered as having a general theoretical meaning, as the values of L/d upon which it has been founded were derived from the assumption of an ellipsoidal shape of the molecules (adjusted so as to fit in with the observed values of the sedimentation constant).